Supersymmetry and the Higgs Sector of the Standard Model

Diploma thesis
in theoretical physics

presented by

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Abstract

A basic supersymmetric theory containing the Higgs sector of the Standard Model is the Minimal Supersymmetric Standard Model. In this theory the bare Higgs masses are known. In order to obtain a preciser phenomenology, corrections must be performed. These corrections come from two different considerations. On the one hand the approach of an effective field theory is taken. This means that a fundamental mass scale Λ *is introduced, at which new physics enters. The low-energy theory will be modified by non renormalizable supersymmetric gauge invariant operators. On the other hand, the Minimal Supersymmetric Standard Model will be influenced by radiative corrections. They are linked to the scale of supersymmetry breaking,* M_{SUSY}*. Modifications to the tree-level Higgs masses arising from these corrections will be calculated.*

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Chapter 1

Introduction

The $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ invariant Lagrangian of the Standard Model (SM) describes massless gauge bosons interacting with massless fermions. In order for mass terms to be generated, one introduces fundamental complex scalar (Higgs) fields ϕ, which couple gauge-invariantly to the gauge bosons via the covariant derivatives

$$\mathcal{D}_\mu \phi \mathcal{D}^\mu \phi^\dagger, \qquad \mathcal{D}_\mu := \partial_\mu + ig \sum_{i=1}^{3} W_\mu^i \frac{\sigma^i}{2} + ig' \frac{Y}{2} B_\mu, \tag{1.1}$$

(where g, g' and $\sigma^i/2$, $Y/2$ and W_μ^i, B_μ are the $SU(2)_L$, $U(1)_Y$ couplings and group generators and gauge fields, respectively) and couple gauge-invariantly to the fermions through the Yukawa coupling of the form

$$-\lambda_Y \left[(\bar{\psi}_L \phi) \psi_R + \bar{\psi}_R (\phi^\dagger \psi_L) \right]. \tag{1.2}$$

This specifies ϕ as a $SU(2)_L$ doublet (weak isospin $T = 1/2$) with weak hypercharge $Y = 1$, i. e.

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1^+ + i\phi_2^+ \\ \phi_1^0 + i\phi_2^0 \end{pmatrix}. \tag{1.3}$$

The problem of generating mass terms in the SM is related to the problem of breaking the gauge symmetry. This is implemented by the mechanism of *spontaneous symmetry breaking*: The Lagrangian remains invariant under the symmetry but the vacuum state is chosen to be invariant. This is exactly what the Higgs fields accomplish, due to an extra term arising in the SM Lagrangian coming from the Higgs self-interaction

$$\mathcal{L}_{Higgs} = -m_H^2 \phi^\dagger \phi - \lambda_H (\phi^\dagger \phi)^2 =: -\mathcal{V}(\phi). \tag{1.4}$$

The scalar potential $\mathcal{V}$ contains the most general $SU(2)_L$ invariant renormalizable (i. e. dimension-four) terms. The dimensionless coupling λ_H must be positive for $\mathcal{V}$ to be positive. In perturbation theory one expands ϕ about the minimum of the scalar potential

$$\left. \frac{\partial \mathcal{V}}{\partial \phi} \right|_{<\phi>} = 0, \tag{1.5}$$

where $<\phi>:=<0\,|\,\phi\,|\,0>$ denotes the vacuum expectation value (VEV). This defines the vacuum state of the theory, which is said to be invariant (non invariant) for a zero (non zero) VEV. The mass parameter m_H^2 controls the spontaneous symmetry breaking: If $m_H^2 > 0$, it would describe the mass of the Higgs particles and $<\phi>= 0$. However, if m_H^2 is taken to be negative[1], then the minimum of $\mathcal{V}$ is shifted and the new condition is

$$<\phi^\dagger\phi>= -\frac{m_H^2}{2\lambda_H} =: \frac{1}{2}v^2. \tag{1.6}$$

This only specifies the magnitude of ϕ, so the symmetry is not yet broken. In choosing a specific solution, i. e. a special direction of ϕ in the isospin space, the symmetry is spontaneously broken, $<\phi>\neq 0$.[2] A valid choice is

$$<\phi>= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \tag{1.7}$$

meaning that $<\phi_1^0>= v$. In order to obtain a meaningful quantum field theory one must shift the fields: $h(x) := \phi_1^0(x) - v$, with $<h>= 0$. The h field is the physical Higgs field after spontaneous symmetry breaking and the Lagrangian — expressed in terms of h — is not invariant. The mass term is $m_h^2 = 2\lambda_H v^2$.

Returning to the ϕ fields, the choice of the minimum as in eq. (1.7) ensures that the vacuum is invariant under $U(1)_{EM}$ and the photon remains massless. This should not be confused with the *Nambu-Goldstone theorem*:

> A Lagrangian $\mathcal{L}(\phi)$ is invariant under some symmetry group G, generated by t^a.[3] The potential is also invariant, meaning that
>
> $$\delta V = \frac{\partial\mathcal{V}}{\partial\phi_i}\delta\phi_i = 0; \qquad i = 1,\ldots,N. \tag{$*$}$$
>
> The equations for the minimum of the potential are
>
> $$\left.\frac{\partial\mathcal{V}}{\partial\phi_i}\right|_{<\phi_i>} = 0 \quad\text{and}\quad \left.\frac{\partial^2\mathcal{V}}{\partial\phi_i\partial\phi_j}\right|_{<\phi_i>} =: \mathcal{M}_{ij}^2 > 0, \tag{$**$}$$
>
> where $\mathcal{M}^2$ is the scalar mass matrix for the fields. Let $<\phi_i>= v_i$ be a solution of these equations. Suppose that some of the generators satisfy $(t^a)_i^k <\phi_k>= 0$, leaving the new vacuum invariant ($a = 1,\ldots,n$). They form a subgroup of G. The remaining $(N-n)$ generators are taken to break the symmetry of the vacuum: $(t^a)_i^k <\phi_k>\neq 0$. They will be denoted by $\tilde{t}^a$. Differentiating eq. ($*$) gives
>
> $$i\omega_a\left(\frac{\partial^2\mathcal{V}}{\partial\phi_i\partial\phi_j}(t^a)_i^k\phi_k + \frac{\partial\mathcal{V}}{\partial\phi_i}(t^a)_i^j\right) = 0. \tag{$***$}$$
>
> Substituting eqs. ($**$) gives $\mathcal{M}_{ij}^2(\tilde{t}^a <\phi>)_i = 0$, which means that for every broken generator there is a zero eigenvalue of $\mathcal{M}^2$. Thus there are $(N-n)$ massless fields, called *Goldstone bosons*.

[1] This corresponds to *tachyon* states.

[2] Note that only scalar fields can have non zero VEV's without violating Lorentz symmetry. This is the motivation for their introduction.

[3] Compare with section 2 of appendix B.

The gauge group for the SM is $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ and one finds

$$\frac{\lambda^a}{2} <\phi>= 0, \qquad \frac{\sigma^i}{2} <\phi>\neq 0, \qquad \frac{Y}{2} <\phi>\neq 0, \tag{1.8}$$

where $\lambda^a/2$, $\sigma^i/2$ and $Y/2$ generate the gauge group ($a = 1, \ldots, 8$; $i = 1, 2, 3$). The Goldstone theorem states that $SU(2)_L \otimes U(1)_Y$ is broken, resulting in four massless Goldstone bosons. $SU(3)_C$ remains unbroken and the corresponding gauge bosons (gluons) remain massless. By construction (eq. (1.7)) the vacuum is invariant under $U(1)_{EM}$

$$Q <\phi>= \left(\frac{\sigma^3}{2} + \frac{Y}{2}\right) <\phi>= 0, \tag{1.9}$$

because of the choice of the non vanishing VEV to be that of the neutral field ϕ_1^0. Thus the symmetry breaking has the structure $SU(2)_L \otimes U(1)_Y \to U(1)_{EM}$ and one is left with three Goldstone bosons (from $SU(2)_L$) and one massless gauge boson (photon) from the remaining $U(1)_{EM}$.

The question remains: how are the masses generated in the SM? The answer is given, when one applies the Goldstone theorem to gauge theories. This is called the *Higgs mechanism*:

> It is possible to parametrize the field ϕ by the introduction of new fields $(\xi_1, \xi_2, \xi_3) = \vec{\xi}$ and h^0
>
> $$\phi = e^{i(\vec{\xi}\cdot\vec{\sigma})/2} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + h^0) \end{pmatrix}, \tag{$\dagger$}$$
>
> where the general $SU(2)_L$ gauge transformation
>
> $$e^{i(\vec{\xi}\cdot\vec{\sigma})/2} = \begin{pmatrix} \cos(\xi/2) + in_3 \sin(\xi/2) & (n_2 + n_1)\sin(\xi/2) \\ -(n_2 - n_1)\sin(\xi/2) & \cos(\xi/2) - in_3 \sin(\xi/2) \end{pmatrix} \tag{$\dagger\dagger$}$$
>
> was used. $\vec{\xi} = \xi\vec{n}$. By inspections h^0 is the physical Higgs field, formerly denoted by h. This allows the three ξ_i fields to be identified as the three Goldstone bosons related to the spontaneous symmetry breaking. They will be employed to give the mass to the physical gauge bosons $W^\pm$ and Z^0.[4] This is achieved by exploiting gauge freedom and setting $\vec{\xi} = 0$ (*unitary gauge*). Thus the Goldstone bosons are 'gauged away' and do not appear in the Lagrangian. The $SU(2)_L$ transformation ($\dagger$) is also linked to a re-definition of the gauge fields W_μ^i, which results in the appearance of mass terms for $W^\pm$ and Z^0 in the Lagrangian . In becoming massive, each gauge boson has received an additional (longitudinal) polarization from the (three) degrees of freedom lost by choosing the unitary gauge.

[4] $W_\mu^\pm := (W_\mu^1 \mp iW_\mu^2)/\sqrt{2}$. The neutral gauge bosons Z_μ^0 and the $U(1)_{EM}$ gauge field A_μ (photon field) are linear combinations of W_μ^3 and B_μ, involving a mixing angle. See footnote on page 34.

For further information on the SM and the Higgs sector consult refs. [1], [2] and [10].

Ideas and concepts of supersymmetry (SUSY) are introduced in the next chapter. An important result will be the formulation of the general gauge invariant supersymmetric Lagrangian in section 2.4.2. The simplest way to incorporate SUSY into the physics of our world is in supersymmetrizing the Lagrangian of the SM. If this is done in the most straightforward manner the resulting theory is called the Minimal Supersymmetric Standard Model (MSSM). It is described in chapter 4. The Higgs sector of the MSSM is presented in section 4.2.2. Being supersymmetric the MSSM resolves the (deep-rooted) complications associated with the Higgs sector of the SM. These complications are referred to as the *hierarchy problem* (HP) and involve the difficulties which arise, if a fundamental scalar Higgs field is included in the SM :

> The mass renormalization gives quadratically divergent corrections to the bare Higgs mass of the form $\delta m_H^2 = g^2\Lambda^2$, where Λ is a cut-off, which indicates the advent of new physics, (e. g. $\Lambda = M_{Planck} \approx 10^{19}$ [GeV]). It is possible to absorb the infinities into the bare mass and obtain the desired values for the physical Higgs masses. This, however, involves an incredible fine tuning of the parameters at every loop order in perturbation theory. In other words: why is the Higgs mass small, and why does it remain small?

SUSY theories[5] circumvent this dilemma due to their special renormalization properties discussed in section 2.5: To each boson loop correction there exists a fermion loop correction of the opposite sign, so that the scalar masses are no longer quadratically divergent, i. e. $\delta m_H^2 = g^2\Lambda_{boson}^2 - g^2\Lambda_{fermion}^2 = 0$. Since SUSY is not an exact symmetry of nature, the cancellations must be incomplete, and the Higgs mass receives contributions that are limited by the extent of the SUSY breaking, controlled by the parameter M_{SUSY}. In order for the HP to remain resolved, it is necessary that the scale of the SUSY breaking does not exceed 1 [TeV], see ref. [3]. It is also considered to be larger than the electroweak scale. The mechanism of SUSY breaking is specified in chapter 3.

The physics at the fundamental mass scale Λ is not specified. It will however manifest itself at energies below Λ through small deviations from the MSSM. They will be described by an effective Lagrangian containing non renormalizable (i. e. dimension ≥ 5) $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ invariant supersymmetric operators. This analysis is in the spirit of ref. [5] , where the calculations of higher dimensional operators is done for the SM. Only next-to-leading order terms in the Lagrangian will be considered and their corrective influence on the bare MSSM Higgs masses is discussed in chapter 5. The range of Λ is taken to be: 3 [TeV] $\leq \Lambda \leq$ 10 [TeV], from ref. [4].[6]

The tree-level Higgs masses will also receive radiative corrections. They are taken from [6] and their influence is again analyzed in chapter 5.

[5]Incidently, there is a different approach to the HP: the Higgs particle is thought to be composed of fermions and a new confining, asymptotically free, non-Abelian gauge interaction is introduced, termed *technicolor*.

[6]Note that this choice of $\Lambda \ll M_{Planck}$ also offers a solution to the HP.

Chapter 2

An Introduction to Supersymmetry

In this chapter the key elements of SUSY are introduced. For a more detailed treatment of the subject and further reading consult references [7] – [11].

2.1 Basic Ideas

SUSY is a symmetry relating bosonic and fermionic fields, meaning that there exist SUSY operators[1] Q, which convert boson states into fermion states (and vice versa):

$$Q\,|B\,\rangle = |F\,\rangle. \tag{2.1}$$

Thus SUSY is a symmetry sensitive to the spin of particles.

Infinitesimal supersymmetric transformations can be expressed in a form analogous to the transformations of the Lorentz group, eq. (B.12)

$$\delta^{SUSY}\Phi = i\bar{\epsilon}Q\Phi. \tag{2.2}$$

The (super-) multiplets Φ span a linear representation of the SUSY algebra generated by Q. The ϵ are the parameters of the transformation. Both Q and ϵ are Majorana spinors (for a summary on spinors see appendix C). The generators themselves are fermionic, so they must satisfy anticommutator relations. It is this special feature of the SUSY generators that give them their unique physical significance[2]: It is possible to extend the Poincaré algebra in a natural way to include these anticommutator relations and thus create a new superalgebra with the structure of a graded Lie algebra

$$\begin{aligned} [\text{even}, \text{even}] &= \text{even} \\ \{\text{odd}, \text{odd}\} &= \text{even} \\ [\text{even}, \text{odd}] &= \text{odd}, \end{aligned} \tag{2.3}$$

[1] In this paper, only the case of one operator will be studied. In the literature this model is referred to as *N=1 SUSY*.

[2] The Coleman-Mandula theorem forbids a nontrivial combination of an internal symmetry group with the Poincaré group. The generators of these groups satisfy commutation relations.

where 'odd' and 'even' denote the fermionic respectively the bosonic character of the generators. This is the only nontrivial way to mix internal and external (i. e. space-time) symmetries and hinges on the concept of spin.

2.2 The Supersymmetry Algebra

The SUSY algebra written with four-component Majorana spinors Q_α reads

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}) \tag{2.4a}$$
$$[M^{\mu\nu}, P^\rho] = -i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu) \tag{2.4b}$$
$$[P^\mu, P^\nu] = 0 \tag{2.4c}$$
$$[M^{\mu\nu}, Q_\alpha] = (\Sigma^{\mu\nu}Q)_\alpha \tag{2.4d}$$
$$[P^\mu, Q_\alpha] = 0 \tag{2.4e}$$
$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C)_{\alpha\beta}P_\mu. \tag{2.4f}$$

Eqs. (2.4a) – (2.4c) are the commutation relations for the Poincaré algebra.[3] Eq. (2.4d) states, that the quantities Q transform as spinors under Lorentz transformations.[4] This also means that the generators of the fermionic sector, i. e. eq. (2.4f), span a representation of the Lie algebra forming the bosonic sector, i. e. the commutation relations (2.4a) – (2.4e).

It is possible to express the anticommutation relations with two-component spinors[5], noting that the Majorana spinor Q_α decomposes into:

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}; \qquad \alpha, \dot\alpha = 1, 2. \tag{2.5}$$

The fermionic sector of the SUSY algebra is then

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \tag{2.6a}$$
$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2P_\mu(\sigma^\mu)_{\alpha\dot\beta}. \tag{2.6b}$$

For the supermultiplets one finds the following properties:

1. All particles belonging to an irreducible representation have the same mass (P^2 is a Casimir operator of the SUSY algebra).

2. The energy $E = P_0$ is always positive or zero.

3. A supermultiplet always contains an equal number of fermionic and bosonic degrees of freedom.

The supersymmetric transformations (2.2) can be cast into two-component notation

$$\delta\Phi = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}\right)\Phi. \tag{2.7}$$

[3] See section 1.1 of appendix B.
[4] See section 1.2 of appendix B.
[5] See appendix D.

It is possible to express the algebra of eq. (2.6b) in terms of commutators, using supersymmetric transformations:

$$\begin{aligned}[\delta_1,\delta_2] &= [i(\epsilon_1 Q+\bar\epsilon_1\bar Q), i(\epsilon_2 Q+\bar\epsilon_2\bar Q)] = -\left(\epsilon_1^\alpha\bar\epsilon_2^{\dot\alpha}+\bar\epsilon_1^{\dot\alpha}\epsilon_2^\alpha\right)\{Q_\alpha,\bar Q_{\dot\alpha}\} \\ &= -2\left(\epsilon_1^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\epsilon_2^{\dot\alpha}-\epsilon_2^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\epsilon_1^{\dot\alpha}\right)P_\mu = 2i\left(\epsilon_1\sigma^\mu\bar\epsilon_2-\epsilon_2\sigma^\mu\bar\epsilon_1\right)\partial_\mu.\end{aligned} \tag{2.8}$$

In performing the above calculations one assumes that the spinorial parameters ϵ^α and $\bar\epsilon^{\dot\alpha}$ are anticommuting (Grassmann) variables

$$\begin{aligned}\{\epsilon^\alpha,\epsilon^\beta\} = \{\epsilon^\alpha,\bar\epsilon^{\dot\beta}\} = \{\bar\epsilon^{\dot\alpha},\bar\epsilon^{\dot\beta}\} = 0 \\ \{\epsilon^\alpha,Q^\beta\} = \{\epsilon^\alpha,\bar Q^{\dot\beta}\} = \{\bar\epsilon^{\dot\alpha},Q^\beta\} = \{\bar\epsilon^{\dot\alpha},\bar Q^{\dot\beta}\} = 0,\end{aligned} \tag{2.9}$$

and it was used that

$$P_\mu = -i\partial_\mu. \tag{2.10}$$

This establishes the fact that the transformation properties of the supermultiplets Φ under supersymmetric transformations specify a representation of the SUSY algebra, i. e.

$$[\delta_1,\delta_2]\Phi = 2i\left(\epsilon_1\sigma^\mu\bar\epsilon_2-\epsilon_2\sigma^\mu\bar\epsilon_1\right)\partial_\mu\Phi. \tag{2.11}$$

2.3 Superfields and Superspace

The superfield and superspace formalism allow a natural and compact description of SUSY representations. Furthermore, Lagrangians can be easily constructed in this framework. Originally SUSY was formulated in the Wess-Zumino model [12] using component fields which form a supermultiplet. Both formulations are equivalent (superfields can be constructed from component fields and component fields can be recovered from superfields by power series expansion), but with increasing complexity the Wess-Zumino model becomes intractable.

2.3.1 Superspace

The idea is to extend ordinary space-time to incorporate additional anticommuting spinorial coordinates subject to the Majorana condition:

$$X = (x^\mu, \theta^\alpha), \qquad \alpha = 1,\dots,4. \tag{2.12}$$

A supersymmetric transformation acting on the superspace coordinates induces the following transformation

$$\begin{aligned}x^\mu &\to x^\mu + i\bar\epsilon\gamma^\mu\theta \\ \theta^\alpha &\to \theta^\alpha + \epsilon^\alpha,\end{aligned} \tag{2.13}$$

where ϵ^α ($\alpha = 1, \dots, 4$) is the parameter of the transformation, again an anticommuting Majorana spinor. In two-component notation eqs. (2.13) are written as

$$\begin{aligned} x^\mu &\to x^\mu + i\epsilon\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\epsilon} \\ \theta^\alpha &\to \theta^\alpha + \epsilon^\alpha \\ \bar{\theta}^{\dot{\alpha}} &\to \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}. \end{aligned} \tag{2.14}$$

In the case of space-time translations, the action of a group element $G = \exp(iP^\mu\Delta_\mu)$ on a scalar function $f(x)$ induces a motion in the parameter space of the form

$$x^\mu \to x^\mu + \Delta^\mu. \tag{2.15}$$

In analogy, the SUSY algebra (eqs. (2.4)) is a Lie algebra with anticommuting parameters and the corresponding group element is

$$G(x, \theta, \bar{\theta}) = e^{i(P^\mu\Delta_\mu + \epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})}. \tag{2.16}$$

So eqs. (2.14) represent a motion in the super-parameter space induced by the action of the group element G on a scalar function of superspace (*superfield*).[6]

2.3.2 Superfields

It is possible to introduce superfields $F = F(x, \theta, \bar{\theta})$ such that the SUSY generators Q and $\bar{Q}$ act on F through derivatives only, in analogy with translations. In other words, a representation of the generators is found, containing the partial derivatives[7] ∂_μ, ∂_α and $\bar{\partial}_{\dot{\alpha}}$:

$$\begin{aligned} Q'_\alpha &= -i\left(\partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right) \\ \bar{Q}'_{\dot{\alpha}} &= -i\left(-\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\right). \end{aligned} \tag{2.17}$$

These differential operators satisfy the required algebra of eq. (2.6b), using $P_\mu = -i\partial_\mu$

$$\{Q'_\alpha, \bar{Q}'_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \tag{2.18}$$

By convention, in the literature a different representation of Q and $\bar{Q}$ is used

$$\begin{aligned} Q''_\alpha &= -i\left(\partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right) \\ \bar{Q}''_{\dot{\alpha}} &= -i\left(-\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\right). \end{aligned} \tag{2.19}$$

This however gives the wrong sign in eq. (2.6b)

$$\{Q''_\alpha, \bar{Q}''_{\dot{\alpha}}\} = +2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \tag{2.20}$$

[6] The result of this analysis is presented in eq.(2.22).

[7] For the notation see appendix A.

In the following this convention will be adopted and the symbol Q (and $\bar{Q}$ respectively) will be used to denote the generators and the representation Q'' (and $\bar{Q}''$ respectively). Note that with

$$\bar{Q}^{\dot{\alpha}} = -i\left(\bar{\partial}^{\dot{\alpha}} + i\theta^{\beta}(\sigma^{\mu})_{\beta\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\partial_{\mu}\right), \tag{2.21}$$

one finds

$$\epsilon Q + \bar{\epsilon}\bar{Q} = -i\left(\epsilon^{\alpha}\partial_{\alpha} + \bar{\epsilon}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} + i(\epsilon\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\epsilon})\partial_{\mu}\right). \tag{2.22}$$

This states the expected result that a supersymmetric transformation generates a translation $i(\epsilon\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\epsilon})$ in x^{μ}-space and a translation ϵ and $\bar{\epsilon}$ of the coordinates θ and $\bar{\theta}$, see discussion after eq. (2.16).

A general superfield $F(x,\theta,\bar{\theta})$ is defined by its expansion in powers of θ and $\bar{\theta}$:

$$\begin{aligned} F(x,\theta,\bar{\theta}) =& f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) \\ &+ \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x). \end{aligned} \tag{2.23}$$

All higher powers of θ and $\bar{\theta}$ vanish. In eq. (2.23)

- f, m, n, d are complex scalar fields,
- ϕ_{α}, $\bar{\chi}^{\dot{\alpha}}$, $\bar{\lambda}^{\dot{\alpha}}$, ψ_{α} are two-component spinors,
- $v_{\mu} = -\frac{1}{2}(\bar{\sigma}_{\mu})^{\dot{\alpha}\alpha}v_{\alpha\dot{\alpha}}$ is a complex vector field.

This gives 16 real bosonic components (two for each scalar, eight for v_{μ}) and 16 real fermionic components (four for each spinor). The superfield F transforms under supersymmetric transformations as

$$\delta F = i(\epsilon Q + \bar{\epsilon}\bar{Q})F. \tag{2.24}$$

The transformation laws for the component fields is obtained by matching the terms of δF with the corresponding θ and $\bar{\theta}$ powers of the components. Consider a superfield F with no $\bar{\theta}$ components, containing only a complex scalar field z, a two-component spinor ψ_{α} and a second complex scalar field f, i. e.

$$F(x,\theta) = z(x) + \theta\psi(x) + \theta\theta f(x). \tag{2.25}$$

Transforming F and using eq. (2.22) yields

$$\delta F = \left(\epsilon^{\alpha}\partial_{\alpha} - i\theta\sigma^{\mu}\bar{\epsilon}\partial_{\mu}\right)F = \epsilon\psi + 2\theta\epsilon f - i(\theta\sigma^{\mu}\bar{\epsilon})\partial_{\mu}z + \frac{i}{2}\theta\theta(\sigma^{\mu}\bar{\epsilon})^{\alpha}\partial_{\mu}\psi_{\alpha}, \tag{2.26}$$

which gives the transformations of the component fields

$$\begin{aligned} \delta^{\text{comp}}z &= \epsilon\psi \\ \delta^{\text{comp}}\psi_{\alpha} &= 2f\epsilon_{\alpha} - i(\sigma^{\mu}\bar{\epsilon})_{\alpha}\partial_{\mu}z \\ \delta^{\text{comp}}f &= \frac{i}{2}\left((\partial_{\mu}\psi)\sigma^{\mu}\bar{\epsilon}\right). \end{aligned} \tag{2.27}$$

This is the starting point in the Wess-Zumino model, where eqs. (2.27) are obtained by the general transformation rule

$$\begin{aligned}\delta^{\text{comp}}(\text{scalar field}) &= O^{\text{scalar}}(\text{spinor field})\\ \delta^{\text{comp}}(\text{spinor field}) &= O^{\text{spinor}}(\text{scalar field}),\end{aligned} \tag{2.28}$$

and the O are two spin-$\frac{1}{2}$ operators containing an infinitesimal spinorial parameter ϵ. One finds in (four-component notation) that

$$[\delta_1^{\text{comp}}, \delta_2^{\text{comp}}] \begin{pmatrix} z \\ \psi \\ f \end{pmatrix} = -2i(\bar{\epsilon}_2 \gamma^\mu \epsilon_1)\partial_\mu \begin{pmatrix} z \\ \psi \\ f \end{pmatrix}, \tag{2.29}$$

(compare with eq. (2.11)). The ψ fields describe spin-$\frac{1}{2}$ particles and the scalar fields z their supersymmetric partners (scalar fermions or *sfermions*). The scalar fields f are so-called *auxiliary* fields, used to close the algebra off-shell, i. e. when the spinor ψ doesn't satisfy the equation of motion $(i\gamma^\mu\partial_\mu - m) = 0$. The f have algebraic equations of motion, do not propagate and do not bring new on-shell degrees of freedom.

It is always possible to construct a superfield corresponding to a component supermultiplet, by starting with any member of the multiplet and acting on it with the transformation (2.24) iteratively until the multiplet is closed

$$F = z + \delta z + \delta^2 z + \dots \tag{2.30}$$

Superfields encompass supermultiplets in a very natural manner and give a straightforward recipe for the generation of new representations of SUSY from old ones, because linear combinations of superfields are again superfields and similarly, products of superfields are again superfields. A general superfield F consists of nine component fields, which form a reducible representation. It is possible to eliminate the redundant components by imposing covariant constraints of the form $\mathcal{D}F = 0$ or $\bar{\mathcal{D}}F = 0$, where $\mathcal{D}$ and $\bar{\mathcal{D}}$ are *covariant derivatives*. These constraints solely reduce the number of components in a superfield and have no dynamical content. So the superfield formalism shifts the problem of finding SUSY representations to that of finding appropriate constraints. The covariant derivatives are constructed to transform covariantly under supersymmetric transformations

$$\mathcal{D}_\alpha(\delta F) = \delta(\mathcal{D}_\alpha F) \quad \text{and} \quad \bar{\mathcal{D}}_{\dot{\alpha}}(\delta F) = \delta(\bar{\mathcal{D}}_{\dot{\alpha}} F), \tag{2.31}$$

which corresponds to

$$\{Q_\alpha, \mathcal{D}_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \mathcal{D}_\beta\} = \{Q_\alpha, \bar{\mathcal{D}}_{\dot{\beta}}\} = \{\bar{Q}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 0. \tag{2.32}$$

The covariant derivatives satisfying eqs. (2.32) are

$$\begin{aligned}\mathcal{D}_\alpha &= \partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\\ \bar{\mathcal{D}}_{\dot{\alpha}} &= \bar{\partial}_{\dot{\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu.\end{aligned} \tag{2.33}$$

Comparing with eqs. (2.17) one finds that $Q'_\alpha = -i\mathcal{D}_\alpha$, $\bar{Q}'_{\dot\alpha} = i\bar{\mathcal{D}}_{\dot\alpha}$ and that

$$\begin{aligned}\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} &= -2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu \\ \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0.\end{aligned} \tag{2.34}$$

This definition of the covariant derivatives can be viewed as a natural generalization of the derivatives ∂_μ, ∂_α and $\bar\partial_{\dot\alpha}$ to the superfield formalism. Note that $\partial_\mu F$, $\mathcal{D}_\alpha F$ and $\bar{\mathcal{D}}_{\dot\alpha} F$ are still superfields, whereas $\partial_\alpha F$ and $\bar\partial_{\dot\alpha} F$ are not. One also finds that $\mathcal{D}^3 = \bar{\mathcal{D}}^3 = 0$.

2.3.3 Chiral Superfields

Chiral superfields are defined by the condition

$$\bar{\mathcal{D}}_{\dot\alpha}\phi = 0. \tag{2.35}$$

This constraint is easily solved in terms of the coordinates $y^\mu := x^\mu - i\theta\sigma^\mu\bar\theta$, by noting that $\bar{\mathcal{D}}_{\dot\alpha} y^\mu = \bar{\mathcal{D}}_{\dot\alpha}\theta = 0$. Thus a chiral superfield is a function of y and θ only:

$$\phi(y,\theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y). \tag{2.36}$$

The factors $\sqrt{2}$ and -1 are introduced by convention. ϕ contains a complex scalar field z, a left-handed Weyl spinor ψ_α and a second complex scalar field f. This defines the chiral multiplet (z, ψ, f). One can expand ϕ around x:

$$\begin{aligned}\phi(x,\theta,\bar\theta) =& z(x) + \sqrt{2}\theta\psi(x) - \theta\theta f(x) - i(\theta\sigma^\mu\bar\theta)\partial_\mu z(x) \\ &+ \frac{i}{\sqrt{2}}\theta\theta(\partial_\mu\psi(x)\sigma^\mu\bar\theta) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^\mu\partial_\mu z(x).\end{aligned} \tag{2.37}$$

The supersymmetric differential operators expressed in terms of y, θ and $\bar\theta$ are

$$\begin{aligned}Q_\alpha &= -i\partial_\alpha \\ \bar{Q}_{\dot\alpha} &= -i\left(-\bar\partial_{\dot\alpha} - 2i(\theta\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial y^\mu}\right).\end{aligned} \tag{2.38}$$

For the superfield $\phi^\dagger$ with the constraint $\mathcal{D}_\alpha\phi^\dagger = 0$ and $\bar{y}^\mu := x^\mu + i\theta\sigma^\mu\bar\theta$, the corresponding equations are

$$\begin{aligned}\phi^\dagger(\bar y,\bar\theta) =& z^*(\bar y) + \sqrt{2}\bar\theta\bar\psi(\bar y) - \bar\theta\bar\theta f^*(\bar y) \\ \phi^\dagger(x,\theta,\bar\theta) =& z^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \bar\theta\bar\theta f^*(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu z^*(x) \\ &- \frac{i}{\sqrt{2}}\bar\theta\bar\theta\left(\theta\sigma^\mu\partial_\mu\bar\psi(x)\right) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^\mu\partial_\mu z^*(x),\end{aligned} \tag{2.39}$$

and in terms of $\bar y$, θ and $\bar\theta$ the operators are

$$\begin{aligned}Q_\alpha &= -i\left(\partial_\alpha + 2i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial\bar y^\mu}\right) \\ \bar{Q}_{\dot\alpha} &= i\bar\partial_{\dot\alpha}.\end{aligned} \tag{2.40}$$

The components of ϕ transform as (compare with eqs. (2.27))

$$\begin{aligned} \delta z &= \sqrt{2}\epsilon\psi \\ \delta\psi_\alpha &= -\sqrt{2}f\epsilon_\alpha - \sqrt{2}i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu z \\ \delta f &= -\sqrt{2}i(\partial_\mu\psi\sigma^\mu\bar{\epsilon}). \end{aligned} \tag{2.41}$$

Note that f transforms into a total derivative under δ. It is possible to define the components of a chiral superfield ϕ using

$$\begin{aligned} \phi|_{\theta=\bar{\theta}=0} &= z \\ \mathcal{D}_\alpha\phi|_{\theta=\bar{\theta}=0} &= \psi_\alpha \\ \mathcal{D}\mathcal{D}\phi|_{\theta=\bar{\theta}=0} &= f. \end{aligned} \tag{2.42}$$

Note that $\mathcal{D}\mathcal{D}$ acts as a chiral projection operator.

If ϕ is a chiral superfield, then ϕ^n is also a chiral superfield. A real constant c can also be considered as a chiral superfield.

2.3.4 Vector Superfields

Vector superfields are real superfields:

$$V(x,\theta,\bar{\theta}) = V(x,\theta,\bar{\theta})^\dagger. \tag{2.43}$$

Their power series expansion reads

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &+ \frac{i}{2}\theta\theta[M(x)+iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x)-iN(x)] \\ &+ i\theta\theta\bar{\theta}[\bar{\lambda}(x) + \frac{i}{2}\partial_\mu\chi(x)\sigma^\mu] - i\bar{\theta}\bar{\theta}\theta[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)] \\ &+ \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(x) - \frac{1}{2}\partial^\mu\partial_\mu C(x)]. \end{aligned} \tag{2.44}$$

This defines a vector multiplet containing 8 bosons (the real scalars C, D, M, N and the four components of v_μ) and 8 fermions (the two-component spinors χ_α and λ_α). The special form of eq. (2.44) allows the introduction of a supersymmetric gauge transformation[8]

$$V \to V + \phi + \phi^\dagger, \tag{2.45}$$

where ϕ is a chiral superfield. One finds that under this gauge transformation

$$\begin{aligned} C &\to C + 2Re(z) \\ \chi &\to \chi - i\sqrt{2}\psi \\ \bar{\chi} &\to \bar{\chi} + i\sqrt{2}\bar{\psi} \\ M + iN &\to M + iN + 2if \\ v_\mu &\to v_\mu - i\partial_\mu(z - z^*) \\ \lambda &\to \lambda \\ \bar{\lambda} &\to \bar{\lambda} \\ D &\to D. \end{aligned} \tag{2.46}$$

[8]See section 2.4.

It is possible to eliminate C, M, N, χ and one component of v_μ with an appropriate choice of the chiral superfield ϕ, i. e. by choosing z, ψ, f. This gauge, called *Wess-Zumino gauge*, reduces the vector multiplet to $(v_\mu, \lambda, \bar\lambda, D)$, which contains four bosons and four fermions. In supersymmetric gauge theories v_μ will correspond to the (vector) gauge bosons, λ to their supersymmetric partners, called *gauginos*[9] and the scalars D are again auxiliary fields. The components transform as

$$\begin{aligned}
\delta v_\mu &= i\epsilon\sigma^\mu\bar\lambda - i\lambda\sigma^\mu\bar\epsilon + \partial_\mu\chi\epsilon + \bar\epsilon\partial_\mu\bar\chi \\
\delta\lambda &= iD\epsilon - \frac{1}{2}(\sigma^\mu\bar\sigma^\nu\epsilon)(\partial_\mu v_\nu - \partial_\nu v_\mu) \\
\delta\bar\lambda &= -iD\bar\epsilon - \frac{1}{2}(\sigma^\mu\bar\sigma^\nu\bar\epsilon)(\partial_\mu v_\nu - \partial_\nu v_\mu) \\
\delta D &= \epsilon\sigma^\nu\partial_\nu\bar\lambda + \partial_\nu\lambda\sigma^\nu\bar\epsilon.
\end{aligned} \tag{2.47}$$

Again the auxiliary field D transforms into a total derivative.

Powers of V are also vector superfields and in the Wess-Zumino gauge one finds:

$$\begin{aligned}
V_{WZ} &= \theta\sigma^\mu\bar\theta v_\mu + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta D \\
V_{WZ}^2 &= (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)v_\mu v_\nu = \frac{1}{2}\theta\theta\bar\theta\bar\theta v^\mu v_\mu \\
V_{WZ}^n &= 0, \quad n \geq 3.
\end{aligned} \tag{2.48}$$

2.4 Supersymmetric Gauge Theories

2.4.1 The Supersymmetric Lagrangian

In order to obtain supersymmetric Lagrangians from superfields, one needs to recall the transformation law for superfields under supersymmetric transformations, eq. (2.24): $\delta F = i(\epsilon Q + \bar\epsilon\bar Q)F$. The $\theta\theta\bar\theta\bar\theta$ component of δF results from the $\epsilon\sigma^\mu\bar\theta\partial_\mu$ part of ϵQ acting on the $\theta\theta\bar\theta$ component of F, respectively the $\theta\sigma^\mu\bar\epsilon\partial_\mu$ part of $\bar\epsilon\bar Q$ acting on the $\bar\theta\bar\theta\theta$ component of F. So the $\theta\theta\bar\theta\bar\theta$ components of a superfield always transform with a total derivative; they are taken to be the supersymmetric Lagrangian. The corresponding action will be invariant under supersymmetric transformations, because all terms with derivatives vanish after integration over d^4x.

Chiral Superfields

For chiral superfields ϕ_n, the $\theta\theta$ components transform like a total derivative (eq. (2.27)). The most general supersymmetric Lagrangian for chiral multiplets is of the form

$$\mathcal{L} = \sum_{n\geq 1} [a_n\phi_n]_{\theta\theta} + h.c., \tag{2.49}$$

[9]Sometimes the two-component spinors λ and $\bar\lambda$ are combined to form a single four-component Majorana spinor also denoted by λ.

where $[\dots]_{\theta\theta}$ indicates that only the $\theta\theta$ component is retained. The linear term ($n = 1$) will only be included for spontaneous breaking of SUSY (see section 3.1). All terms with powers $n \geq 4$ are dropped, because they lead to non renormalizable field theories. The quadratic terms have a mass-coupling and the cubic terms a dimensionless coupling. They are combined into

$$\mathcal{W}(\phi) := \frac{1}{2} m_{ij}\phi_i\phi_j + \frac{1}{3}\lambda_{ijk}\phi_i\phi_j\phi_k, \tag{2.50}$$

where the function $\mathcal{W}$ is called *superpotential*[10] and depends only on ϕ_i and not on $\phi_i^\dagger$.[11] Its $\theta\theta$ component reads

$$\begin{aligned}[\mathcal{W}(\phi_i)]_{\theta\theta} = &-\frac{1}{2} m_{ij}(z_i f_j + z_j f_i + \psi_i\psi_j)\\ &-\frac{1}{3}\lambda_{ijk}(z_i z_j f_k + z_j z_k f_i + z_k z_i f_j + z_i\psi_j\psi_k + z_j\psi_k\psi_i + z_k\psi_i\psi_j).\end{aligned} \tag{2.51}$$

Up to now no terms containing derivatives of the fields (i. e. kinetic terms) have been found. They can only be constructed from $\phi_i^\dagger\phi_i$, which is not a chiral superfield. The $\theta\theta\bar\theta\bar\theta$ component is

$$\begin{aligned}\mathcal{L}^{kin} = [\phi_i^\dagger\phi_i]_{\theta\theta\bar\theta\bar\theta} =& f_i^* f_i + (\partial_\mu z_i)(\partial^\mu z_i^*) + \frac{i}{2}\left(\psi_i\sigma^\mu(\partial_\mu\bar\psi_i) - (\partial_\mu\psi_i)\sigma^\mu\bar\psi_i\right)\\ &-\frac{1}{4}\partial_\mu(z_i\partial^\mu z_i^* + z_i^*\partial^\mu z_i).\end{aligned} \tag{2.52}$$

The last term is a derivative and will be dropped. The most general renormalizable supersymmetric Lagrangian involving only chiral superfields is

$$\mathcal{L}^{ch} = \mathcal{L}^{kin} + [\mathcal{W}(\phi)]_{\theta\theta} + [\overline{\mathcal{W}}(\phi^\dagger)]_{\bar\theta\bar\theta}. \tag{2.53}$$

Using integrals over superspace (see appendix D) eq. (2.53) can be recast:

$$\mathcal{L}^{ch} = \int d^2\theta d^2\bar\theta\phi_i^\dagger\phi_i + \int d^2\theta\mathcal{W}(\phi) + \int d^2\bar\theta\,\overline{\mathcal{W}}(\phi^\dagger). \tag{2.54}$$

The auxiliary fields f can be eliminated from $\mathcal{L}^{ch}$ by virtue of their equation of motion, i. e. by varying the Lagrangian with respect to f

$$\frac{\partial\mathcal{L}^{ch}}{\partial f} = 0 \quad\Longleftrightarrow\quad f_i^* = m_{ij}z_j + \lambda_{ijk}z_j z_k. \tag{2.55}$$

The component form of the Lagrangian is then

$$\begin{aligned}\mathcal{L}^{ch}_{comp} =& (\partial_\mu z_i)(\partial^\mu z_i^*) + \frac{i}{2}\left(\psi_i\sigma^\mu(\partial_\mu\bar\psi_i) - (\partial_\mu\psi_i)\sigma^\mu\bar\psi_i\right) - \frac{1}{2}m_{ij}(\psi_i\psi_j + \bar\psi_i\bar\psi_j)\\ &- \lambda_{ijk}z_i\psi_j\psi_k - \lambda_{ijk}^* z_i^*\bar\psi_j\bar\psi_k - \mathcal{V},\end{aligned} \tag{2.56}$$

[10]Note that the computation of $\phi_i\phi_j$ and $\phi_i\phi_j\phi_k$ should be done in the $x, \theta, \bar\theta$ variables.

[11]The superpotential for $\phi^\dagger$ is denoted by $\overline{\mathcal{W}}$.

where the scalar potential $\mathcal{V}$ was introduced

$$\mathcal{V} = \sum_i \left| \frac{\partial \mathcal{W}(z)}{\partial z_i} \right|^2 = \sum_i |m_{ij} z_j + \lambda_{ijk} z_j z_k|^2, \tag{2.57}$$

and m_{ij} was taken to be real. The fermion mass terms and the Yukawa couplings can be expressed as

$$\begin{aligned} &-(\frac{1}{2} m_{ij} \psi_i \psi_j + \lambda_{ijk} z_i \psi_j \psi_k + \frac{1}{2} m_{ij} \bar{\psi}_i \bar{\psi}_j + \lambda^*_{ijk} z^*_i \bar{\psi}_j \bar{\psi}_k) \\ &= -\frac{1}{2} \frac{\partial^2 \mathcal{W}(z)}{\partial z_i \partial z_j} \psi_i \psi_j - \frac{1}{2} \frac{\partial^2 \overline{\mathcal{W}}(z^*)}{\partial z^*_i \partial z^*_j} \bar{\psi}_i \bar{\psi}_j . \end{aligned} \tag{2.58}$$

Vector Superfields

The vector multiplet contains the necessary fields to construct gauge theories. There exists however the same problem as in the case with chiral superfields, namely the impossibility of generating kinetic terms from the vector superfields V — compare with eqs. (2.48). They can be obtained by acting on V with the covariant derivatives. The required form for the kinetic terms comes from the chiral superfields

$$\begin{aligned} W_\alpha &= -\frac{1}{4} (\bar{\mathcal{D}} \bar{\mathcal{D}}) \mathcal{D}_\alpha V \\ \overline{W}_{\dot\alpha} &= -\frac{1}{4} (\mathcal{D} \mathcal{D}) \bar{\mathcal{D}}_{\dot\alpha} V. \end{aligned} \tag{2.59}$$

Both W_α and $\overline{W}_{\dot\alpha}$ are gauge invariant under the transformation (2.45). In the Wess-Zumino gauge and using the variables y, θ and $\bar{y}$, $\bar\theta$ they read

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha(y) + \theta_\alpha D(y) + \frac{i}{2} (\theta \sigma^\mu \bar\sigma^\nu)_\alpha F_{\mu\nu}(y) - \theta\theta \left(\sigma^\mu \partial_\mu \bar\lambda(y) \right)_\alpha \\ \overline{W}_{\dot\alpha} &= i\bar\lambda_{\dot\alpha}(\bar{y}) + \bar\theta_{\dot\alpha} D(\bar{y}) - \frac{i}{2} (\sigma^\mu \bar\sigma^\nu \bar\theta)_{\dot\alpha} F_{\mu\nu}(\bar{y}) - \bar\theta\bar\theta \left(\partial_\mu \lambda(\bar{y}) \sigma^\mu \right)_{\dot\alpha} , \end{aligned} \tag{2.60}$$

where the field strengths are

$$F_{\mu\nu}(y) = \partial_\mu v_\nu(y) - \partial_\nu v_\mu(y). \tag{2.61}$$

The supersymmetric gauge invariant Lagrangian involving only vector superfields is found to be

$$\begin{aligned} \mathcal{L}^{vect} &= \frac{1}{4} [W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{4} [\overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha}]_{\bar\theta\bar\theta} \\ &= \frac{i}{2} (\lambda \sigma^\mu (\partial_\mu \bar\lambda) - (\partial_\mu \lambda) \sigma^\mu \bar\lambda) + \frac{1}{2} D^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \end{aligned} \tag{2.62}$$

where

$$[W^\alpha W_\alpha]_{\theta\theta} = 2i\lambda \sigma^\mu \partial_\mu \bar\lambda + D^2 - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} . \tag{2.63}$$

In order to obtain the last relation eq. (D.26) was used.

2.4.2 Supersymmetric Gauge Theories

In a next step the gauge invariant interactions of chiral and vector multiplets is considered. This will be a generalization of the formalism of gauge theories to SUSY and will give rise to the most general renormalizable supersymmetric gauge theory, describing the interactions of a set of chiral superfields ϕ^i and a set of vector superfields V^a. As in the case of non supersymmetric theories, the ϕ^i transform according to an arbitrary representation **N** of the gauge group G and the V^a belong to the adjoint representation of G, where $a = 1, \ldots, \dim G$.

Under global transformations of the symmetry group G, the chiral superfields transform as (see eq. (B.24))

$$\phi'^i = \left(e^{i\Lambda^a T^a}\right)^i_{\ j} \phi^j, \tag{2.64}$$

or infinitesimally,

$$\delta\phi^i = i\Lambda^a (T^a)^i_{\ j}\phi^j. \tag{2.65}$$

The matrices T^a are the hermitian generators of G in the representation **N**, defined by the chiral superfields ϕ. The transformation parameters Λ^a are real constants, i. e. chiral superfields. So eq. (2.64) is a superfield equation. The Lagrangian for the chiral multiplet $\mathcal{L}^{ch}$ (eqs. (2.53) or (2.54)) contains two terms, $\mathcal{L}^{kin}$ and $\mathcal{W}$. The kinetic terms are naturally invariant under the transformations (2.64), because $\phi^\dagger \to \phi^\dagger \exp(-i\Lambda^a T^a)$. The requirement of invariance of the superpotential imposes constraints on the individual terms comprising it, i. e. they must be independantly invariant.

If one wants to gauge the transformation (2.64), a few disorders have to be remedied. The local transformation parameters $\Lambda^a(x)$ are no longer superfields and have to be promoted to chiral superfields in order for the ϕ'^i to remain chiral superfields. This changes the transformation law for $\phi_i^\dagger$ to

$$\phi_i'^\dagger = \phi_j^\dagger \left(e^{-i\Lambda^{a\dagger} T^a}\right)^j_{\ i}, \tag{2.66}$$

which destroys the invariance of $\mathcal{L}^{kin}$. To restore gauge invariance it is necessary to introduce vector superfields V^a. This situation is totally analogous to non supersymmetric gauge theories, where gauge fields are introduce to reinstate (local) invariance. Gauge invariance is restored by requiring the transformation law for e^V to be

$$e^V \to e^{i\Lambda^\dagger} e^V e^{-i\Lambda}, \tag{2.67}$$

and by redefining

$$\mathcal{L}^{kin}_{g.i.} = \int d^2\theta d^2\bar{\theta} (\phi^\dagger e^V \phi) = \left[\phi^\dagger e^V \phi\right]_{\theta\theta\bar{\theta}\bar{\theta}}, \tag{2.68}$$

where the matrix notation

$$\Lambda_{ij} := \Lambda^a T^a_{ij}, \qquad V_{ij} := V^a T^a_{ij} \tag{2.69}$$

was used and the indices omitted. The transformation (2.67) corresponds to an infinitesimal transformation

$$\delta V = -i(\Lambda - \Lambda^\dagger), \tag{2.70}$$

which is a gauge transformation in the spirit of eq. (2.45). So the Wess-Zumino gauge is a choice of local gauge in supersymmetric gauge theories. Then the kinetic terms are

$$\begin{aligned}\left[\phi_i^\dagger (e^V)^i{}_j \phi^j\right]_{\theta\theta\bar\theta\bar\theta} =& (\mathcal{D}_\mu z)_i^* (\mathcal{D}^\mu z)^i + \frac{i}{2}(\psi^i \sigma^\mu (\mathcal{D}_\mu \bar\psi)_i - (\mathcal{D}_\mu \psi)^i \sigma^\mu \bar\psi_i) + f_i^* f^i \\ &+ \frac{i}{\sqrt{2}}(\bar\psi_i \bar\lambda^a)(T^a)^i{}_j z^j - \frac{i}{\sqrt{2}} z_i^* (T^a)^i{}_j (\lambda^a \psi^j) \\ &+ \frac{1}{2} D^a z_i^* (T^a)^i{}_j z^j,\end{aligned} \tag{2.71}$$

with the covariant derivatives

$$\begin{aligned}(\mathcal{D}_\mu z)^i &= \partial_\mu z^i + \frac{i}{2}(T^a)^i{}_j z^j v_\mu^a \\ (\mathcal{D}_\mu \psi)^i &= \partial_\mu \psi^i + \frac{i}{2}(T^a)^i{}_j \psi^j v_\mu^a.\end{aligned} \tag{2.72}$$

Having solved all the difficulties for $\mathcal{L}^{ch}$, the next step is to consider the kinetic terms for the vector superfields, i. e. the chiral superfields W_α and $\overline{W}_{\dot\alpha}$. They are also no longer invariant under the gauge transformations (2.67) if the gauge group is non-Abelian ($[\Lambda, V] \neq 0$). The correct generalizations are obtained by the redefinition

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}} e^{-V} \mathcal{D}_\alpha e^V \tag{2.73a}$$

$$\overline{W}_{\dot\alpha} = -\frac{1}{4}\mathcal{D}\mathcal{D} e^{-V} \bar{\mathcal{D}}_{\dot\alpha} e^V. \tag{2.73b}$$

The transformations are

$$\begin{aligned}W_\alpha &\to e^{-i\Lambda} W_\alpha e^{i\Lambda} \\ \overline{W}_{\dot\alpha} &\to e^{-i\Lambda} \overline{W}_{\dot\alpha} e^{i\Lambda}.\end{aligned} \tag{2.74}$$

In the Wess-Zumino gauge and using the matrix notation

$$v_\mu = v_\mu^a T^a, \qquad \lambda = \lambda^a T^a, \qquad D = D^a T^a, \tag{2.75}$$

eq. (2.73a) reads

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + \frac{i}{2}(\theta\sigma^\mu\bar\sigma^\nu)_\alpha F_{\mu\nu}(y) - \theta\theta\left(\sigma^\mu \mathcal{D}_\mu \bar\lambda(y)\right)_\alpha, \tag{2.76}$$

where the field strengths and covariant derivatives are taken to be

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu v_\nu - \partial_\nu v_\mu + \frac{i}{2}[v_\mu, v_\nu] \\ \mathcal{D}_\mu \bar\lambda^{\dot\alpha} &= \partial_\mu \bar\lambda^{\dot\alpha} + \frac{i}{2}[v_\mu, \bar\lambda^{\dot\alpha}].\end{aligned} \tag{2.77}$$

The hermitian generators T^a, taken in the adjoint representation, are normalized as

$$\mathrm{tr}(T^aT^b) = \kappa\delta^{ab}, \qquad \kappa > 0. \tag{2.78}$$

With this convention, the structure constants f^{abc}

$$[T^a, T^b] = if^{abc}T^c \tag{2.79}$$

are totally antisymmetric. In component form eqs. (2.77) read

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu v^a_\nu - \partial_\nu v^a_\mu - \frac{1}{2}f^{abc}v^b_\mu v^c_\nu \\ \mathcal{D}_\mu\bar{\lambda}^a &= \partial_\mu\bar{\lambda}^a - \frac{1}{2}f^{abc}v^b_\mu\bar{\lambda}^c. \end{aligned} \tag{2.80}$$

Coupling constants g are introduced by rescaling $V \to 2gV$. One then has

$$\mathrm{tr}[W^\alpha W_\alpha]_{\theta\theta} = 4g^2\kappa\left(2i\lambda^a\sigma^\mu(\mathcal{D}_\mu\bar{\lambda})^a + D^aD^a - \frac{1}{2}F^a_{\mu\nu}F^{a\mu\nu} - \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\right), \tag{2.81}$$

where the trace is taken over the gauge group indices. The invariant kinetic Lagrangian involving the vector multiplet reads

$$\begin{aligned} \mathcal{L}^{vect}_{g.i.} &= \frac{1}{16g^2\kappa}\left(\mathrm{tr}[W^\alpha W_\alpha]_{\theta\theta} + \mathrm{tr}[\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}]_{\bar{\theta}\bar{\theta}}\right) \\ &= \frac{i}{2}(\lambda^a\sigma^\mu(\mathcal{D}_\mu\bar{\lambda})^a - (\mathcal{D}_\mu\lambda)^a\sigma^\mu\bar{\lambda}^a) + \frac{1}{2}D^aD^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \end{aligned} \tag{2.82}$$

The complete supersymmetric gauge invariant Lagrangian is then

$$\begin{aligned} \mathcal{L}^{g.i.} = \mathcal{L}^{ch}_{g.i.} + \mathcal{L}^{vect}_{g.i.} &= \left[\phi^\dagger_i(e^{2gV})^i{}_j\phi^j\right]_{\theta\theta\bar{\theta}\bar{\theta}} + [\mathcal{W}(\phi^i)]_{\theta\theta} + [\overline{\mathcal{W}}(\phi^\dagger_i)]_{\bar{\theta}\bar{\theta}} \\ &+ \frac{1}{16g^2\kappa}\left(\mathrm{tr}[W^\alpha W_\alpha]_{\theta\theta} + \mathrm{tr}[\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}]_{\bar{\theta}\bar{\theta}}\right), \end{aligned} \tag{2.83}$$

or in components

$$\begin{aligned} \mathcal{L}^{g.i.}_{comp} =& (\mathcal{D}_\mu z)^*_i(\mathcal{D}^\mu z)^i + \frac{i}{2}(\psi^i\sigma^\mu(\mathcal{D}_\mu\bar{\psi})_i - (\mathcal{D}_\mu\psi)^i\sigma^\mu\bar{\psi}_i) + f^*_if^i \\ &+ i\sqrt{2}g(\bar{\psi}_i\bar{\lambda}^a)(T^a)^i{}_jz^j - i\sqrt{2}gz^*_i(T^a)^i{}_j(\lambda^a\psi^j) + gD^az^*_i(T^a)^i{}_jz^j \\ &- \frac{1}{2}\frac{\partial^2\mathcal{W}(z^k)}{\partial z^i\partial z^j}\psi^i\psi^j - \frac{1}{2}\frac{\partial^2\overline{\mathcal{W}}(z^*_k)}{\partial z^*_i\partial z^*_j}\bar{\psi}_i\bar{\psi}_j - \frac{\partial\mathcal{W}(z^k)}{\partial z^i}f^i - \frac{\partial\overline{\mathcal{W}}(z^*_k)}{\partial z^*_i}f^*_i \\ &+ \frac{i}{2}\lambda^a\sigma^\mu(\mathcal{D}_\mu\bar{\lambda})^a - \frac{i}{2}(\mathcal{D}_\mu\lambda)^a\sigma^\mu\bar{\lambda}^a + \frac{1}{2}D^aD^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \end{aligned} \tag{2.84}$$

with

$$\begin{aligned} (\mathcal{D}_\mu z)^i &= \partial_\mu z^i + ig(T^a)^i{}_jz^jv^a_\mu \\ (\mathcal{D}_\mu\psi)^i &= \partial_\mu\psi^i + ig(T^a)^i{}_j\psi^jv^a_\mu \\ (\mathcal{D}_\mu\lambda)^a &= \partial_\mu\lambda^a - gf^{abc}v^b_\mu\bar{\lambda}^c \\ F^a_{\mu\nu} &= \partial_\mu v^a_\nu - \partial_\nu v^a_\mu - gf^{abc}v^b_\mu v^c_\nu. \end{aligned} \tag{2.85}$$

The equations of motion of the auxiliary fields f^i and D^a are

$$\frac{\partial \mathcal{L}}{\partial f^i} = 0 \quad \Longleftrightarrow \quad f_i^* = \frac{\partial \mathcal{W}(z^k)}{\partial z^i} \tag{2.86a}$$

$$\frac{\partial \mathcal{L}}{\partial D^a} = 0 \quad \Longleftrightarrow \quad D^a = -g z_i^* (T^a)^i{}_j z^j. \tag{2.86b}$$

When substituted into the Lagrangian one finds

$$\begin{aligned}\mathcal{L}^{g.i.}_{comp} =&(\mathcal{D}_\mu z)_i^* (\mathcal{D}^\mu z)^i + \frac{i}{2}\psi^i \sigma^\mu (\mathcal{D}_\mu \bar{\psi})_i - \frac{i}{2}(\mathcal{D}_\mu \psi)^i \sigma^\mu \bar{\psi}_i \\ &+ i\sqrt{2}g(\bar{\psi}_i \bar{\lambda}^a)(T^a)^i{}_j z^j - i\sqrt{2} g z_i^* (T^a)^i{}_j (\lambda^a \psi^j) \\ &+ \frac{i}{2}\lambda^a \sigma^\mu (\mathcal{D}_\mu \bar{\lambda})^a - \frac{i}{2}(\mathcal{D}_\mu \lambda)^a \sigma^\mu \bar{\lambda}^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \\ &- \frac{1}{2}\frac{\partial^2 \mathcal{W}(z^k)}{\partial z^i \partial z^j}\psi^i \psi^j - \frac{1}{2}\frac{\partial^2 \overline{\mathcal{W}}(z_k^*)}{\partial z_i^* \partial z_j^*}\bar{\psi}_i \bar{\psi}_j - \mathcal{V}(z^i, z_j^*),\end{aligned} \tag{2.87}$$

where the scalar potential is

$$\begin{aligned}\mathcal{V}(z^i, z_j^*) &= \sum_i |f^i|^2 + \frac{1}{2}\sum_a (D^a)^2 \\ &= \sum_i \left|\frac{\partial \mathcal{W}}{\partial z^i}\right|^2 + \frac{1}{2}g^2 \sum_a \left(z_i^* (T^a)^i{}_j z^j\right)^2 \geq 0,\end{aligned} \tag{2.88}$$

and (with no summation implied)

$$\mathcal{W}(\phi) := \frac{1}{2}m_{ij}\phi^i \phi^j + \frac{1}{3}\lambda_{ijk}\phi^i \phi^j \phi^k. \tag{2.89}$$

2.5 Renormalization

In considering radiative (loop) corrections, supersymmetric gauge theories differ greatly from ordinary renormalizable field theories. This is due to the fact that many divergences usually present in non supersymmetric theories are canceled.[12]

The mechanisms of renormalization are those of any quantum field theory: The unobservable divergences are absorbed into renormalization constants. This must be performed for the (chiral and vector) superfields, the Yukawa couplings and the mass parameters of the superpotential and for the gauge couplings:

$$\begin{aligned}\phi^i &= (Z_\phi^{1/2})^i{}_j \phi^j_{(ren)} \\ V^a &= (Z_V^{1/2})^{ab} V^b_{(ren)} \\ \lambda_{ijk} &= (Z_\lambda^{-1})^{lmn}_{i'j'k'} \lambda^{(ren)}_{lmn} \\ m_{ij} &= (Z_m^{-1})^{lm}_{i'j'}[m^{(ren)}_{lm} - \delta m_{lm}] \\ g^a &= (Z_V^{-1/2})^{ab} g^b_{(ren)}.\end{aligned} \tag{2.90}$$

[12] An example of this effect was given in chapter 1, where the quadratic divergences associated with the scalar (Higgs) masses got canceled, solving the hierarchy problem of the SM.

The Lagrangian is then expressed in terms of the renormalized quantities and recast in a way, such that all the infinities are collected in so called *counter terms*

$$\mathcal{L} = \mathcal{L}_{ren} + \mathcal{L}_{c.t.}. \tag{2.91}$$

There is however a theorem in supersymmetric gauge theories — called 'non-renormalization theorem' — with far-reaching consequences:

- The parameters of the superpotential are not renormalized by higher-loop corrections. They receive no contribution at all, not even finite ones. This means that

$$\begin{aligned} (Z_\lambda)_{ijk}^{lmn} &= \delta_i^{\,l}\delta_j^{\,m}\delta_k^{\,n} \\ (Z_m)_{ij}^{lm} &= \delta_i^{\,l}\delta_j^{\,m} \\ \delta m_{ij} &= 0, \end{aligned} \tag{2.92}$$

  and

$$\begin{aligned} [\lambda_{ijk}\phi^i\phi^j\phi^k]_{(ren)} &= \lambda_{ijk}\phi^i\phi^j\phi^k \\ [m_{ij}\phi^i\phi^j]_{(ren)} &= m_{ij}\phi^i\phi^j. \end{aligned} \tag{2.93}$$

- Only the wave-function renormalization constants Z_ϕ and Z_V survive and are logarithmically divergent. Thus SUSY has no quadratic divergences.

- At any order in perturbation theory one gets a cancellation of the terms arising from different Feynman diagrams[13]: Every loop diagram has a corresponding supersymmetric counterpart, associated with the supersymmetric particle. Because fermion loops carry a minus sign (Fermi-Dirac statistic), they cancel the scalar (boson) loops.

- Supersymmetric field theories can be finite to all orders in perturbation theory.

[13] The loop integrations are taken over $d^4kd^2\theta d^2\bar{\theta}$.

Chapter 3

Spontaneous Breaking of Supersymmetry

The particles observed in nature show no sign of a degeneracy between fermions and bosons. Hence, SUSY, if it is to be relevant to our reality, must be broken. The possible implementations are the same as for the SM: The breaking can be spontaneous or explicit. In the SM the former possibility is chosen, as described in chapter 1. Explicit breaking would involve the introduction of new non invariant terms into the Lagrangian. Thus there is no mechanism behind the breaking and the whole approach seems quite ad hoc. In this chapter two viable methods of spontaneous SUSY breaking are presented. They however lead to a model which is not allowed by phenomenology. So one has to resort to the less satisfactory explicit breaking of SUSY. Recall that SUSY is a global symmetry. It is natural to ask what the consequences of gauged, i. e. local, SUSY would be. The answer is profound. One arrives at a new and highly sophisticated[1] theory automatically encompassing gravity, called supergravity[2] (SUGRA). Within this theory, ordinary SUSY is an effective theory resulting at low energies. Thus the origin of the explicit SUSY breaking terms is explained.

3.1 The Problem of Supersymmetry Breaking

The breaking of a symmetry is always related to the scalar potential and the VEV's of the fields. For SUSY the scalar potential is given in eq. (2.88) and only scalar fields can have non zero VEV's, $< z^i > \neq 0$, resulting in spontaneous breaking of gauge symmetry and the Higgs mechanism. Recall from section 2.2 that the vacuum energy $< \mathcal{V} >$ is greater than or equal to zero. The vacuum state $|\Omega>$ is supersymmetric (i. e. is invariant under SUSY) if

$$Q_\alpha |\Omega> = Q^\dagger_\alpha |\Omega> = 0, \tag{3.1}$$

[1]This is unfortunately also the case for the formalism, which is — prohibitively — complex.
[2]See appendix F for some general remarks on supergravity and unification.

in which case the (supersymmetric) vacuum has zero energy. In order for SUSY to be spontaneously broken, the vacuum energy

$$<\mathcal{V}>=<f^i f_i^*>+\frac{1}{2}<D^a D^a>, \tag{3.2}$$

must be positive. This is achieved with

$$<f^i>\neq 0 \qquad \text{and/or} \qquad <D^a>\neq 0. \tag{3.3}$$

Note that as a consequence, for broken SUSY, the transformations of eqs. (2.41) and (2.47) now read

$$\begin{aligned} <\delta\psi^i> &= -\sqrt{2}<f^i>\epsilon \neq 0 \\ <\delta\lambda^a> &= i<D^a>\epsilon \neq 0. \end{aligned} \tag{3.4}$$

There are two models related to these realization of SUSY breaking.

1. *D-type breaking or the Fayet-Iliopoulos mechanism*: The idea is, that if the theory has an Abelian $U(1)$ gauge invariance, then the Lagrangian can include a term linear in the gauge superfield V, called Fayet-Iliopoulos term: $\mathcal{L}_{FI} = g\sum_a \xi^a [V^a]_{\theta\theta\bar\theta\bar\theta}$. This introduces a new D-term into the overall Lagrangian of the form $[V^a_{WZ}]_{\theta\theta\bar\theta\bar\theta} = (1/2)D^a$. The new equation of motion reads

$$D^a = -g(z_i^*(T^a)^i{}_j z^j + \xi^a), \tag{3.5}$$

so $<D^a>\neq 0$ is always possible for $<z^i>=0$.

2. *F-type breaking or the O'Raifeartaigh model*: It involves the introduction of a linear term[3] $a_i\phi^i$ in the superpotential $\mathcal{W}$. This generates new f-terms in the scalar potential. The minimal equations for $\mathcal{V}$, with the assumption $<D^a>=0$, forces them to have non zero VEV's. The situation can be characterized as follows

$$\begin{aligned} &<z^i>\neq 0 \text{ and } <f^i>=0 \text{ breaks gauge symmetry,} \\ &<z^i>=0 \text{ and } <f^i>\neq 0 \text{ breaks SUSY,} \\ &<z^i>\neq 0 \text{ and } <f^i>\neq 0 \text{ breaks both symmetries.} \end{aligned}$$

The problem of spontaneous breaking of SUSY is related to the existence of a so called mass formula relating the scalar, fermion and boson masses. This mass formula is a feature of supersymmetric theories and is valid for all possible vacua, breaking or non-breaking. It reads

$$\text{Str }\mathcal{M}^2 := 3\text{tr }\mathcal{M}_1^2 - 2\text{tr }\mathcal{M}_{1/2}^2 + \text{tr }\mathcal{M}_0^2 = 0, \tag{3.6}$$

where $\mathcal{M}_1^2$, $\mathcal{M}_{1/2}^2$ and $\mathcal{M}_0^2$ are respectively the squared mass matrices of the spin 1, 1/2 (two-component spinors) and 0 (real scalars) fields of the theory. Eq.

[3] This contribution must be invariant under the gauge group of the theory to allow for a supersymmetric gauge invariant model. Thus ϕ^i must be invariant, i. e. a gauge singlet.

(3.6) is however badly broken phenomenologically, because the boson masses cannot be sufficiently heavier than the fermion masses. To resolve this dilemma, one has to resort to the explicit breaking of SUSY. The new terms introduced into the supersymmetric Lagrangian should be checked for their renormalization properties. In order for the hierarchy problem to remain solved, the breaking terms should not introduce quadratic divergences. They are called *soft breaking terms* and involve scalar mass terms of the form $z_i^* z^i$, $(z^i z^j + z_i^* z_j^*)$ and gaugino mass terms λ^a. Their form[4] is calculated in ref. [7]

$$\begin{aligned}\mathcal{L}_{soft} =&(M)_i^{\ j} z^i z_j^* + \frac{1}{2}[(\mu)_{ij} z^i z^j + (\mu^*)^{ij} z_i^* z_j^*] \\ &+ \frac{1}{2}[\Delta^{ab}\lambda^a\lambda^b + \bar{\Delta}^{ab}\bar{\lambda}^a\bar{\lambda}^b].\end{aligned} \tag{3.7}$$

3.2 Supersymmetry Breaking and Supergravity

As mentioned above, the mechanism of spontaneous SUSY breaking can only be allowed to occur, if SUSY is itself a local symmetry and is thus put onto equal footing with the (internal) gauge symmetries. When the Goldstone theorem is applied to spontaneous SUSY breaking, it will produce the usual Goldstone bosons but in addition also their supersymmetric fermionic partners, the *Goldstinos*. The Higgs mechanism for the Goldstone bosons is analogous to the SM case, but now also the gauginos can be gauged away, giving mass to some, as yet, unspecified particle. In order to continue the discussion of this so called *super-Higgs effect*, the theory of supergravity should be (very briefly) explored.

3.2.1 Elements of Supergravity

The discussion will be limited to four-dimensional $N = 1$ SUGRA. Ref. [8] gives a very thorough introduction to the required formalism of curved superspace. The superspace coordinates are denoted by $z = (x, \theta, \bar{\theta})$. As in the case of curved four dimensional space-time, two sets of indices are needed to specify local Lorentz frames and general coordinate frames.[5] The Lorentz indices are denoted by $A, B, \ldots$, and transform under the structure group, i. e. the Lorentz group. A Lorentz index contains the four-vector index a and the spinor indices α and $\dot{\alpha}$. The Einstein indices $M, N, \ldots$ transform under general coordinate transformations (GCT) and contain the components m, μ and $\dot{\mu}$. The relation of the two coordinate systems is given by the vielbein $E_A^{\ M}$. This specifies the geometry of superspace. Physics enters through the hypothesis of invariance of the physical laws under general coordinate transformations[6] $z'^M = z^M + \xi^M(z)$ and local Lorentz transformations (LLT), i. e. rotations of the vielbein $E'^{\ M}_A = [\delta_A^{\ B} + L_A^{\ B}]E_B^{\ M}$. The ξ^M parameterize the GCT and the Lorentz generators $L_A^{\ B}$ have three irreducible components, $L_b^{\ a}$, $L_\beta^{\ \alpha}$ and $L_{\dot{\alpha}}^{\ \dot{\beta}}$. The ξ can

[4] Here the most general $SU(2) \otimes U(1)$ invariant form is presented and will be used later in chapter 4. The most general soft breaking terms are not considered.

[5] A fact related to the equivalence principle of general relativity, which states, that at every point in space-time a local inertial coordinate system can be set up.

[6] These are now local supersymmetric transformation. Compare with eqs. (2.14).

also be written with Lorentz indices using the inverse vielbein, $\xi^A = E_M^{\ A}\xi^M$. Note that both ξ and $L_A^{\ B}$ are functions of superspace, i. e. are superfields containing various component fields. Their lowest superfield components $\theta = \bar\theta = 0$ are, respectively, x-space GCT $\xi^m(x)$, gauged x-space supersymmetry transformations $\xi^\mu(x)$, $\bar\xi^{\dot\mu}(x)$ and x-space LLT $L_a^{\ b}(x)$. Every component of the vielbein is also a superfield. Using a special combination of GCT and LLT (i. e. choosing a special gauge), it is possible to eliminate some of the component fields of $E_A^{\ M}$. The lowest superfield components are then

$$E_A^{\ M}(z)\big|_{\theta=\bar\theta=0} = \begin{pmatrix} e_a^{\ m}(x) & -\frac{1}{2}\psi_a^{\ \mu}(x) & -\frac{1}{2}\bar\psi_a^{\ \dot\mu}(x) \\ 0 & \delta_\alpha^{\ \mu} & 0 \\ 0 & 0 & \delta_{\dot\alpha}^{\ \dot\mu} \end{pmatrix}. \tag{3.8}$$

These remaining fields describe the spin-2 *graviton* ($e_a^{\ m}$) and the spin-$\frac{3}{2}$ *gravitino* ($\psi_a^{\ \mu}, \bar\psi_a^{\ \dot\mu}$). Above procedure can also be applied to the connection $\phi_{MA}^{\quad B}$ (tensor valued 1-form defined by Cartans first structural equation) and the resulting non-zero component fields are

$$\phi_{mA}^{\quad B}(z)\big|_{\theta=\bar\theta=0} =: \omega_{mA}^{\quad B}(x). \tag{3.9}$$

As in the case of general relativity, it is possible to express $\omega_{mA}^{\quad B}(x)$ in terms of the vielbein, i. e. as a function of $e_a^{\ m}(x)$ and $\psi_a^{\ \mu}(x)$, thus identifying these fields as the dynamical variables of the theory. The torsion T is defined as the covariant derivative of the vielbein and the curvature tensor is given in terms of the connection by virtue of Cartans second structural equation

$$R = d\phi + \phi \wedge \phi. \tag{3.10}$$

It is a tensor valued 2-form. The number of components in the torsion and curvature is very large and a set of special constraints is required to reduce it. The torsion and the curvature also satisfy Bianchi identities. One has to solve these identities subject to the constraints to find the reduced set of fields. The result is, that all components of the torsion and curvature can be expressed in terms of three superfields: a chiral superfield S, a hermitian vector superfield $G_{\alpha\dot\alpha}$ and a chiral superfield $W_{\alpha\beta\gamma}$. The lowest components of $W_{\alpha\beta\gamma}$ can be expressed in terms of $e_m^{\ a}$, $\psi_m^{\ \alpha}$ and $\bar\psi_m^{\ \dot\alpha}$. This is however not possible for S and $G_{\alpha\dot\alpha}$, so their lowest components define two additional fields $M(x)$ (complex scalar field) and $b_m(x)$ (real vector field). These two new fields turn out to be the auxiliary fields. The SUGRA multiplet is then $(e, \psi, \bar\psi, M, b)$. Besides the SUGRA multiplet, one needs to introduce the matter fields. In curved superspace the condition of a chiral superfield is

$$\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0. \tag{3.11}$$

The new curved-space covariant derivatives are defined as

$$\begin{aligned} \mathcal{D}_M U^A &= \partial_M U^A + (-1)^{f(M)f(B)}\phi_{MB}^{\quad A}U^B \\ \mathcal{D}_B U^A &= E_B^{\ M}\mathcal{D}_M U^A, \end{aligned} \tag{3.12}$$

where $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha})$ and U represents an arbitrary tensor field carrying Lorentz indices and

$$f = \begin{cases} 0, & \text{for vector indices;} \\ 1, & \text{for spinor indices.} \end{cases} \tag{3.13}$$

The components of Φ are defined in analogy to eqs. (2.42)

$$\begin{aligned} A &= \Phi|_{\theta=\bar\theta=0} \\ \chi_\alpha &= \frac{1}{\sqrt{2}}\,\mathcal{D}_\alpha\Phi|_{\theta=\bar\theta=0} \\ F &= -\frac{1}{4}\,\mathcal{D}^\alpha\mathcal{D}_\alpha\Phi|_{\theta=\bar\theta=0}\,, \end{aligned} \tag{3.14}$$

and carry Lorentz indices. New fermionic coordinates Θ^α are introduced, such that the expansion coefficients of chiral superfields are precisely the covariant components, i. e.

$$\Phi = A(x) + \sqrt{2}\Theta^\alpha\chi_\alpha(x) + \Theta^\alpha\Theta_\alpha F(x). \tag{3.15}$$

The Θ coordinates are called *chiral superspace coordinates.* For the construction of the chiral superfield W_α from the vector superfields V one generalizes eq. (2.42)

$$W_\alpha = -\frac{1}{4}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\mathcal{D}_\alpha V, \qquad \bar{W}_{\dot\alpha} = -\frac{1}{4}(\mathcal{D}^\alpha\mathcal{D}_\alpha - 8R^\dagger)\bar{\mathcal{D}}_{\dot\alpha}V, \tag{3.16}$$

where $(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)$ and $(\mathcal{D}\mathcal{D} - 8R^\dagger)$ are the curved superspace chiral projection operators.

The next step is to construct the Lagrangians. The invariant SUGRA Lagrangian is

$$\mathcal{L}_{SUGRA} = -\frac{6}{\kappa^2}\int d^2\Theta\mathscr{E}R + \text{h.c.} \tag{3.17}$$

The gravitational coupling $\kappa^2 = 8\pi G_N$ is set to one. The so called chiral density $\mathscr{E}$ is needed for chiral Lagrangians to be invariant. R and $\mathscr{E}$ can be expanded in terms of Θ and their components are functions of the SUGRA multiplet. Inserting these expressions into eq. (3.17) yields

$$\begin{aligned} \mathcal{L}_{SUGRA} = &-\frac{1}{2}e\mathscr{R} - \frac{1}{3}eM^*M + \frac{1}{3}eb^a b_a \\ &+ \frac{1}{2}e\varepsilon^{klmn}(\bar\psi_k\bar\sigma_l\tilde{\mathcal{D}}_m\psi_n - \psi_k\sigma_l\tilde{\mathcal{D}}_m\bar\psi_n), \end{aligned} \tag{3.18}$$

where e is the determinant of the vielbein component e_m^a and $\mathscr{R}$ is the scalar curvature

$$\mathscr{R} = e_a{}^n e_b{}^m(\partial_n\omega_m{}^{ab} - \partial_m\omega_n{}^{ab} + \omega_m{}^{ac}\omega_{nc}{}^b - \omega_n{}^{ac}\omega_{mc}{}^b). \tag{3.19}$$

The covariant derivative is defined as

$$\tilde{\mathcal{D}}_n\psi_m{}^\alpha = \partial_n\psi_m{}^\alpha + \psi_m{}^\beta\omega_{n\beta}{}^\alpha. \tag{3.20}$$

From eq. (3.18) one can see that $\mathcal{L}_{SUGRA}$ contains the Einstein Lagrangian for the gravitational field. It also contains a kinetic (Rarita-Schwinger) term for the spin-$\frac{3}{2}$ gravitino. A curved superspace generalization of the chiral Lagrangian in eq. (2.54) is considered next. The kinetic term can be recast using $\int d^2\theta d^2\bar{\theta} = \int d^2\theta(-1/8)\bar{D}\bar{D}$. The replacements

$$\theta \to \Theta, \qquad d^2\theta \to d^2\Theta 2\mathscr{E}, \qquad \bar{D}\bar{D} \to (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R), \tag{3.21}$$

give the required curved superspace form. $\mathcal{L}_{SUGRA}$ is then added, yielding the chiral SUGRA Lagrangian for the matter multiplet. It can be written in the form

$$\mathcal{L}^{ch}_{SUGRA} = \int d^2\Theta 2\mathscr{E}\left(-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\Omega(\Phi,\Phi^\dagger) + P(\Phi)\right) + \text{h.c.}, \tag{3.22}$$

where $\Omega(\Phi,\Phi^\dagger) = \Phi_i^\dagger\Phi_i + c_i\Phi_i + c_i^*\Phi_i^\dagger - 3$ is the superspace kinetic energy, and $P(\Phi) = d + a_i\Phi_i + (1/2)m_{ij}\Phi_i\Phi_j + (1/3)\lambda_{ijk}\Phi_i\Phi_j\Phi_k$ is the superpotential. The c and d terms arise from shifts in the superfields Φ. They are zero in flat superspace. The most general chiral SUGRA Lagrangian is of the form

$$\mathcal{L} = \frac{1}{\kappa^2}\int d^2\Theta 2\mathscr{E}\left(\frac{3}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\exp\left[-\frac{\kappa^2}{3}K(\Phi,\Phi^\dagger)\right] + \kappa^2 P(\Phi)\right) + \text{h.c.}, \tag{3.23}$$

where $K(\Phi,\Phi^\dagger)$ is a hermitian function of the superfields, called the *Kähler potential*, and $P(\Phi)$ is the superpotential. An expansion in κ^2 includes the terms $\mathcal{L}_{SUGRA}$ and $\mathcal{L}^{ch}_{SUGRA}$, for $K = \Omega$.

The construction of invariant models is very technical and lengthy.[7] The form of the scalar potential is however quite straight-forward.

3.2.2 The Super-Higgs Effect and the Effective Low-Energy Lagrangian

The most general gauge invariant SUGRA Lagrangian $\mathcal{L}^{g.i.}_{SUGRA}$ contains a scalar potential of the form

$$\mathcal{V} = e^G(G_iG^i - 3), \tag{3.24}$$

and a gravitino mass term

$$-e^{(G/2)}(\psi_a\sigma^{ab}\psi_b + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b), \tag{3.25}$$

where $G = K + \log P + \log P^\dagger$ is the shifted Kähler potential and

$$G_i := \frac{\partial G}{\partial \Phi^i}, \qquad G^i := \frac{\partial G}{\partial \Phi_i^\dagger}. \tag{3.26}$$

[7] The most general gauge invariant SUGRA Lagrangian written in terms of the component fields fills two pages in ref. [8].

The exponential factor arises from the (Weyl) rescaling of the ${e_m}^a$ fields, required to bring the Einstein Lagrangian contained in $\mathcal{L}^{g.i.}_{SUGRA}$ into the canonical form. This rescaling also implies a redefinition of the fermion fields. In order for the scalar potential of eq. (3.24) to be comparable with the SUSY scalar potential one chooses

$$G(\Phi, \Phi^\dagger) = \frac{\Phi_i^\dagger \Phi^i}{M^2} + \log \left| \frac{\mathcal{W}}{M^3} \right|^2 , \tag{3.27}$$

where $\mathcal{W}$ is the superpotential of section 2.4 and the gravitational coupling is reintroduced

$$M := \kappa^{-1} = \frac{M_{Planck}}{\sqrt{8\pi}}. \tag{3.28}$$

With this choice the scalar potential is

$$\mathcal{V} = \exp\left(\frac{\Phi_i^\dagger \Phi^i}{M^2}\right) \left\{ \left| \frac{\partial \mathcal{W}}{\partial \Phi^i} + \frac{\Phi_i^\dagger \mathcal{W}}{M^2} \right|^2 - \frac{3}{M^2} |\mathcal{W}|^2 \right\}. \tag{3.29}$$

In the large-M limit this reduces to the scalar potential of global SUSY.

Minimizing the most general scalar potential of eq. (3.24) at non zero VEV's of the scalar components A^i gives a condition for the VEV of the Kähler potential

$$<G> \neq 0, \tag{3.30}$$

which spontaneously breaks SUSY and the gravitino acquires a mass

$$m_{3/2} = < e^{G/2} > . \tag{3.31}$$

This is the Higgs mechanism for fermions, the super-Higgs effect. Local SUSY is spontaneously broken and the previously massless gravitinos (gauginos) acquire a mass, their extra polarization degrees of freedom being supplied by absorption of the Goldstinos associated with the breaking of SUSY. The gravitino mass is also associated with the cosmological constant Λ^{cos}, which can be said to represent the vacuum energy of the universe. In SUGRA either the gravitino is massless and the appearance of its mass term is due to the cosmological constant being non zero, or one sets $\Lambda^{cos} = 0$ and the physical mass of the gravitino is obtained. The latter viewpoint states, that in spontaneously broken local SUSY it is possible to have zero vacuum energy $< \mathcal{V} >$. An example of this can be seen in eq. (3.29), where the negative contributions allow the scalar potential to be zero. Recall that with broken global SUSY, $<\mathcal{V}>$ is always positive.

Note that the local supersymmetric scalar potential of eq. (3.29) differs from the global version due to additional terms related to the gravitational coupling $\kappa = 1/M$. At low energy physics their effect is negligible unless some of the scalar fields A^i take VEV's that are $\mathcal{O}(M)$. These fields are said to belong to the so called 'hidden sector'. An example of spontaneous symmetry breaking in the hidden sector is given in the Polonyi model. This is the simplest realization of the super-Higgs effect. The model comprises only one chiral multiplet and the superpotential is

$$\mathcal{W}_h(A) = m^2(A+\beta), \tag{3.32}$$

where A is the scalar field in the hidden sector and β is a model-dependent parameter. From eq. (3.29) the scalar potential becomes

$$\mathcal{V}^P = \exp\left(\frac{|A|^2}{M^2}\right)\left\{\left|m^2 + \frac{m^2(A+\beta)A^*}{M^2}\right|^2 - \frac{3m^4}{M^2}|A+\beta|^2\right\}. \tag{3.33}$$

The condition of a zero cosmological constant is imposed, so $<\mathcal{V}>= 0$. The corresponding minimum constraints are

$$A = (\sqrt{3}-1)M \quad \text{and} \quad \beta = (2-\sqrt{3})M. \tag{3.34}$$

Inserting these values into eqs. (3.27) and (3.31) yields

$$m_{3/2} = \frac{m^2}{M}e^{(2-\sqrt{3})}. \tag{3.35}$$

In a next step the model is enlarged to include observable light scalars, denoted by z^i, with masses $m_{z^i} \ll M$, which are required by low energy phenomenology. The superpotential gets a new term from this 'observable sector'

$$\widehat{\mathcal{W}}(A, z^i) = \mathcal{W}_h(A) + \mathcal{W}_o(z^i). \tag{3.36}$$

This expression is again inserted into eq. (3.29) and yields the scalar potential $\widehat{\mathcal{V}}$, which is then minimized ensuring that Λ^{cos} is zero. A gravitino mass term is found similar to that in eq. (3.35), with additional model-dependent constants.

If one considers a low energy effective theory in the limit $M \to \infty$ (i. e. with gravity decoupled) but $m_{3/2} \ll M$ held fixed, then one receives an effective scalar potential of the form

$$\widehat{\mathcal{V}} \to \mathcal{V}_{eff} = \mathcal{V}_\Lambda + \mathcal{V}_o + \mathcal{V}', \tag{3.37}$$

where $\mathcal{V}_\Lambda$ contains model-dependent terms and can be identified as the cosmological constant of the theory, $\mathcal{V}_o$ is the usual scalar potential of global SUSY with the superpotential $\mathcal{W}_o$. One finds that $\mathcal{V}'$ contains soft breaking terms for the light scalars z^i. Thus

$$\mathcal{L}^{eff}_{SUGRA} = \mathcal{L}_{SUSY} - \mathcal{V}_{soft}, \tag{3.38}$$

meaning that the spontaneous breaking of SUGRA manifests itself at low energies only in the appearance of explicit soft breaking terms. The resulting (gravitationally induced) mass for the light scalars is $m_{3/2}$ and therefore squarks and sleptons are split from their known superpartners. The gravitino mass gives the scale of SUSY breaking and can be maximally $m_{3/2} = 1$ [TeV]. The intermediate scale of the mass parameter m of the hidden sector is then

$$m \approx \sqrt{Mm_{3/2}} \approx 10^{11}\ [\text{GeV}]. \tag{3.39}$$

Chapter 4

Phenomenology of Supersymmetry

Up to now only the theoretical details of SUSY have been discussed. In order to have a phenomenology of SUSY it is necessary to build a supersymmetric theory, containing the SM fermions and bosons. In this model the Higgs sector of the SM will receive important modifications.

4.1 The Minimal Supersymmetric Standard Model

In nature no evidence for the existence of supersymmetric particles has been found. So the particle content of the MSSM must be doubled. The gauge bosons are partnered by gauginos, the fermions by sfermions. The superfields, their components and their $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ quantum numbers are given in tables 4.1 and 4.2. The Lagrangian for the MSSM will be constructed

Table 4.1: MSSM matter fields for one generation: Left-handed squark and quark doublets, left-handed up and down antisquarks and antiquarks, left-handed slepton and lepton doublets, left-handed antisleptons and antileptons.

Superfield	Component fields	Quantum number
Q	$z_Q = \begin{pmatrix} z_u \\ z_d \end{pmatrix}$, $q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(3, 2, \frac{1}{3})$
U^c	z_U^c, u_L^c	$(\bar{3}, 1, -\frac{4}{3})$
D^c	z_D^c, d_L^c	$(\bar{3}, 1, \frac{2}{3})$
L	$z_L = \begin{pmatrix} z_\nu \\ z_e \end{pmatrix}$, $l_L = \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	$(1, 2, -1)$
E^c	$z_E^c,\ e_L^+$	(1,1,2)

Table 4.2: MSSM gauge fields: Winos and W-bosons, binos and B-bosons, gluinos and gluons. Additional physical gauge bosons are the photinos and the photons, the zinos and the Z-bosons (see footnote on page 3).

Superfield	Component fields	Quantum number
W	$(\lambda_W^+, \lambda_W^-, \lambda_W^3)$, $(W_\mu^+, W_\mu^-, W_\mu^3)$	$(1,3,0)$
B	λ_B, B_μ	$(1,1,0)$
G	$(\lambda_{G^1}, \ldots, \lambda_{G^8})$, $(G_\mu^1, \ldots, G_\mu^8)$	$(8,1,0)$

from eq. (2.83). The gauge transformations of the chiral superfields are

$$\begin{aligned} Q^A &\to e^{i\Lambda_3} e^{i\Lambda_2} e^{\frac{1}{3} i\Lambda_1} Q^A \\ U_c^A &\to e^{-i\Lambda_3} e^{-\frac{4}{3} i\Lambda_1} U_c^A \\ D_c^A &\to e^{-i\Lambda_3} e^{\frac{2}{3} i\Lambda_1} D_c^A \\ L^A &\to e^{i\Lambda_2} e^{-i\Lambda_1} L^A \\ E_c^A &\to e^{2i\Lambda_1} E_c^A, \end{aligned} \tag{4.1}$$

where the index A runs over the quark and lepton generations and all the terms should carry indices m, n and α, β belonging to the colour and isospin space, respectively. The matrices $\Lambda_{1,2,3}$ are defined as

$$\Lambda_3 = \sum_{a=1}^{8} \Lambda_3^a \frac{\lambda^a}{2}, \quad \Lambda_2 = \sum_{i=1}^{3} \Lambda_2^i \frac{\sigma^i}{2}, \quad \Lambda_1. \tag{4.2}$$

Thus the chiral superfields Λ_3^a, Λ_2^i and the Λ_1 parametrize the transformation. The λ^a are the Gell-Mann matrices and σ^i the Pauli matrices. In a next step the vector superfields are introduced, as described in section 2.4.2:

$$\begin{aligned} V_3 &= \sum_{a=1}^{8} V_3^a \frac{\lambda^a}{2}, && \text{for } SU(3)_C \\ V_2 &= \sum_{i=1}^{3} V_2^i \frac{\sigma^i}{2}, && \text{for } SU(2)_L \\ V_1&, && \text{for } U(1)_Y. \end{aligned} \tag{4.3}$$

The vector multiplets V_3^a, V_2^a and V_1 correspond to (λ_G^a, G_μ^a), (λ_W^i, W_μ^i) and (λ_B, B_μ), respectively. The chiral supersymmetric gauge invariant Lagrangian can be put into the following form:

$$\begin{aligned} \mathcal{L}_{g.i.}^{ch} = \Big[&(Q^A)^\dagger_{m,\alpha} (e^{V_3})^{mn} (e^{V_2})^{\alpha\beta} e^{\frac{1}{3}V_1} (Q^A)_{n,\beta} + (U_c^A)^\dagger_{m,0} (e^{-V_3})^{mn} e^{-\frac{4}{3}V_1} (U_c^A)_{n,0} \\ &+ (D_c^A)^\dagger_{m,0} (e^{-V_3})^{mn} e^{\frac{2}{3}V_1} (D_c^A)_{n,0} + (L^A)^\dagger_{0,\alpha} (e^{V_2})^{\alpha\beta} e^{-V_1} (L^A)_{0,\beta} \\ &+ (E_c^A)^\dagger_{0,0} e^{2V_1} (E_c^A)_{0,0} \Big]_{\theta\theta\bar\theta\bar\theta}. \end{aligned} \tag{4.4}$$

Note the absence of the superpotential in eq. (4.4). This is due to the fact, that gauge invariant combinations of the MSSM chiral superfields — as specified in eq. (2.50) — lead to baryon and lepton number violation and are thus dropped. The invariant Lagrangian involving the kinetic terms arising from the vector superfields is constructed with the chiral multiplets defined in eqs. (2.73): W_3^ρ for $SU(3)_C$, W_2^ρ for $SU(2)_L$ and W_1^ρ for $U(1)_Y$. Thus

$$\begin{aligned}\mathcal{L}_{g.i.}^{vect} =& \frac{1}{8g_s^2}\left(\mathrm{tr}[W_3^\rho W_{3\rho}]_{\theta\theta} + \mathrm{tr}[\overline{W}_{3\dot\rho}\overline{W}_3^{\dot\rho}]_{\bar\theta\bar\theta}\right) \\ &+ \frac{1}{8g^2}\left(\mathrm{tr}[W_2^\rho W_{2\rho}]_{\theta\theta} + \mathrm{tr}[\overline{W}_{2\dot\rho}\overline{W}_2^{\dot\rho}]_{\bar\theta\bar\theta}\right) \\ &+ \frac{1}{16g'^2}\left([W_1^\rho W_{1\rho}]_{\theta\theta} + [\overline{W}_{1\dot\rho}\overline{W}_1^{\dot\rho}]_{\bar\theta\bar\theta}\right).\end{aligned} \tag{4.5}$$

In the SM, quarks and leptons receive their mass through their Yukawa couplings to a ($Y = 1$) Higgs doublet ϕ, see eq. (1.2). In order to break $SU(2)_L \otimes U(1)_Y$ gauge symmetry, the VEV for ϕ is taken to be

$$<\phi>= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h(x)\end{pmatrix}, \tag{4.6}$$

see eq. (1.7). This will, however, only generate masses for the lower member of the fermion doublets, e. g. electrons and down-quarks. The upper components of the doublets are taken to couple to a $Y = -1$ field

$$\bar\phi = \begin{pmatrix}\bar\phi^0\\ \bar\phi^-\end{pmatrix} := i\sigma^2\phi^* = \begin{pmatrix}\phi^{0*}\\ -\phi^-\end{pmatrix} \quad \text{with} \quad <\bar\phi>= \frac{1}{\sqrt{2}}\begin{pmatrix}v+h(x)\\ 0\end{pmatrix}, \tag{4.7}$$

where $\phi^- := \phi^{+*}$. Thus only one Higgs field is needed in the SM. In the supersymmetric case, Yukawa couplings (for the Higgs fields) arise from the superpotential $\mathcal{W}$, which is only a function of the chiral superfields, but not of their complex conjugate. In order to receive mass terms for all the fermions in the MSSM, one needs two independent Higgs superfield, H and $\bar H$ with quantum numbers $(1, 2, -1)$ and $(1, 2, +1)$, respectively. Their scalar components are

$$z_{\bar H} = \begin{pmatrix}z_{\bar H}^+\\ z_{\bar H}^0\end{pmatrix} \quad \text{and} \quad z_H = \begin{pmatrix}z_H^0\\ z_H^-\end{pmatrix}. \tag{4.8}$$

The allowed[1] terms in the superpotential generated by Higgs and matter superfield interactions are

$$\mathcal{W} = \sum_{A,B}\left(\lambda_E^{AB}L^A H E_c^B + \lambda_D^{AB}Q^A H D_c^B + \lambda_U^{AB}Q^A \bar H U_c^B + mH\bar H\right). \tag{4.9}$$

The multiplication of the $SU(2)_L$ doublet superfields (Q^A, L^A, H, $\bar H$) is understood as $\varepsilon_{ij}(L^A)^i H^j$ etc. So in the MSSM the $Y = 1$ Higgs field couples to up-type fermions, in contrast to the SM. Note the introduction of the mass

[1] Supersymmetric and $SU(2)_L \times U(1)_Y$ gauge invariant.

parameter m in eq. (4.9). When the Higgs superfields H and $\bar{H}$ acquire VEV's, the fermion mass matrices arise from

$$\lambda_E^{AB} < H >, \qquad \lambda_D^{AB} < H >, \qquad \lambda_U^{AB} < \bar{H} >, \tag{4.10}$$

exactly as in the SM, and they are still free parameters. The chiral Lagrangian will also receive additional terms due to the introduction of the Higgs fields. Noting that

$$H \to e^{i\Lambda_2} e^{-i\Lambda_1} H, \qquad \bar{H} \to e^{i\Lambda_2} e^{+i\Lambda_1} \bar{H}, \tag{4.11}$$

one finds the contribution to $\mathcal{L}^{ch}_{g.i.}$ of eq. (4.4) of the form

$$\mathcal{L}^{ch}_H = \Big[(H^\dagger)_\alpha (e^{V_2})^{\alpha\beta} e^{-V_1} (H)_\beta + (\bar{H}^\dagger)_\alpha (e^{V_2})^{\alpha\beta} e^{V_1} (\bar{H})_\beta \Big]_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{4.12}$$

Squarks and sleptons must have large enough masses to make their phenomenology consistent with the absence of any experimental evidence of their existence. This can be achieved with the (ad hoc) introduction of the soft breaking terms of eq. (3.7). They are

$$\begin{aligned}
\mathcal{L}_{soft} = & \sum_A \big((m_Q^A)^2 |z_Q^A|^2 + (m_U^A)^2 |z_U^{cA}|^2 + (m_D^A)^2 |z_D^{cA}|^2 + (m_L^A)^2 |z_L^A|^2 \\
& + (m_E^A)^2 |z_E^{cA}|^2 \big) + m_H^2 |z_H|^2 + m_{\bar{H}}^2 |z_{\bar{H}}|^2 + \mu^2_{H\bar{H}} (z_H z_{\bar{H}} + \text{h.c.}) \\
& + \frac{1}{2} m_3 \sum_{a=1}^{8} (\lambda_3^a \lambda_3^a + \bar{\lambda}_3^a \bar{\lambda}_3^a) + \frac{1}{2} m_2 \sum_{i=1}^{3} (\lambda_2^i \lambda_2^i + \bar{\lambda}_2^i \bar{\lambda}_2^i) \\
& + \frac{1}{2} m_1 (\lambda_1 \lambda_1 + \bar{\lambda}_1 \bar{\lambda}_1),
\end{aligned} \tag{4.13}$$

containing scalar and gaugino[2] mass terms.

The complete MSSM Lagrangian is then

$$\mathcal{L}_{MSSM} = \mathcal{L}^{ch}_{g.i.} + [\,\mathcal{W}_{H\bar{H}}\,]_{\theta\theta} + [\,\overline{\mathcal{W}}_{H\bar{H}}\,]_{\bar{\theta}\bar{\theta}} + \mathcal{L}^{vect}_{g.i.} + \mathcal{L}_{soft}. \tag{4.14}$$

4.2 Analysis of the Higgs Sector

The scalar potential $\mathcal{V}$ of the MSSM contains terms arising from the superpotential $\mathcal{W}$ and from the auxiliary fields D of the vector multiplets associated with the $SU(2)_L$ and $U(1)_Y$ groups — compare with eq. (2.88). It also receives contributions from $\mathcal{L}_{soft}$.

$$\mathcal{V}_{MSSM} = \mathcal{V}_{SUSY} + \mathcal{V}_{soft}. \tag{4.15}$$

The vacuum of the theory is given by the minimum of the scalar potential. The VEV's of squarks and sleptons are taken to be zero, so that colour and lepton numbers remain unbroken.[3] Only z_H and $z_{\bar{H}}$ have non zero VEV's, inducing the

[2]Recall the footnote on page 13.

[3]This justifies the elimination of the superpotential (containing only chiral superfields) in the MSSM Lagrangian.

correct gauge symmetry breaking. It is thus permissible to limit ones attention solely on those terms of $\mathcal{V}$ associated with the scalar Higgs fields. In this section the above introduced notation will be abused and the scalar components z_H, $z_{\bar{H}}$ will be denoted by H, $\bar{H}$ respectively.

Before the Higgs sector of the MSSM is explored, it will be useful to first review a general (non supersymmetric) two-Higgs doublet extension of the SM.

4.2.1 A Generic Two-Higgs Doublet Model

Two complex scalar Higgs fields ϕ_1 and ϕ_2 are introduced. They are $SU(2)_L$ doublets with hypercharge $Y = 1$ and have the following components and VEV's:

$$\phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix} \quad \text{and} \quad <\phi_i> = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_i \end{pmatrix}, \quad i = 1, 2. \tag{4.16}$$

It is possible to chose the phases of the Higgs fields, such that the v_i are real and positive. Eq. (4.16) respects $U(1)_{EM}$ gauge symmetry. The most general gauge invariant potential is given by

$$\begin{aligned} \mathcal{V} =& m_{11}^2 \phi_1^\dagger \phi_1 + m_{22}^2 \phi_2^\dagger \phi_2 - [m_{12}^2 \phi_1^\dagger \phi_2 + \text{h.c.}] + \frac{1}{2}\lambda_1 (\phi_1^\dagger \phi_1)^2 \\ &+ \frac{1}{2}\lambda_2 (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \lambda_4 (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1) \\ &+ \left[\frac{1}{2}\lambda_5 (\phi_1^\dagger \phi_2)^2 + \lambda_6 (\phi_1^\dagger \phi_1)(\phi_1^\dagger \phi_2) + \lambda_7 (\phi_2^\dagger \phi_2)(\phi_1^\dagger \phi_2) + \text{h.c.} \right]. \end{aligned} \tag{4.17}$$

In most discussions of two-Higgs doublet models, the terms m_{12}^2, λ_5, λ_6 and λ_7 are taken to be real. This disregards the possibility of CP-violating effects from neutral and charged Higgs boson interactions; ref. [13]. In order to suppress tree-level flavour changing neutral currents — due to specific Higgs-fermion couplings — one can impose a discrete symmetry[4] of the form $\phi_1 \to -\phi_1$, setting $\lambda_6 = \lambda_7 = 0$; ref. [13]. Actually m_{12}^2 should also be zero, but one allows for a soft violation of the discreet symmetry. This results in radiatively generated flavour changing neutral currents. They are however small enough to avoid conflict with the experiments.

The physical fields are obtained by rotation of ϕ_1 and ϕ_2:

$$\begin{aligned} \tilde{\phi}_1 &= \cos\beta \, \phi_1 + \sin\beta \, \phi_2 \\ \tilde{\phi}_2 &= -\sin\beta \, \phi_1 + \cos\beta \, \phi_2, \end{aligned} \tag{4.18}$$

thus introducing a new parameter to the model:

$$\tan\beta = \frac{v_2}{v_1} \qquad \text{or} \qquad \cos\beta = \frac{v_1}{v} \quad \text{and} \quad \sin\beta = \frac{v_2}{v} \qquad \text{with} \quad v := \sqrt{v_1^2 + v_2^2}. \tag{4.19}$$

[4]Note that the Higgs-fermion couplings are model dependent and that the discrete symmetry is used to make a (natural) discrimination.

The new fields have VEV's $<\tilde{\phi}_1>= v/\sqrt{2}$ and $<\tilde{\phi}_2>=0$, as required. Parametrizing the physical fields (as specified in eq. (†) on page 3) and choosing unitary gauge, yields

$$\tilde{\phi}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+h^0) \end{pmatrix} \quad \text{and} \quad \tilde{\phi}_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}(H^0+iA^0) \end{pmatrix}. \tag{4.20}$$

Of the original eight (scalar) degrees of freedom contained in ϕ_1 and ϕ_2, three Goldstone bosons are gauged away, providing the longitudinal polarization for the gauge bosons $W^\pm$ and Z^0. The remaining five degrees of freedom produce the physical Higgs particles: two neutral scalars h^0 and H^0, one neutral pseudoscalar A^0 and one charged scalar H^+ (with two degrees of freedom). Taking $\tilde{\phi}_2^\dagger$ into consideration introduces an additional charged Higgs particle H^-. Thus $H^\pm = -\sin(\beta)\phi_1^\pm + \cos(\beta)\phi_2^\pm$. The inverse rotation of eq. (4.18) is:

$$\begin{aligned} \phi_1 &= \begin{pmatrix} -\sin\beta\, H^+ \\ \frac{1}{\sqrt{2}}\left[v_1 + \cos\beta\, h^0 - \sin\beta\, H^0 - i\sin\beta\, A^0\right] \end{pmatrix} \\ \phi_2 &= \begin{pmatrix} \cos\beta\, H^+ \\ \frac{1}{\sqrt{2}}\left[v_2 + \sin\beta\, h^0 + \cos\beta\, H^0 + i\cos\beta\, A^0\right] \end{pmatrix}. \end{aligned} \tag{4.21}$$

If these values of ϕ_1 and ϕ_2 are inserted into $\mathcal{V}$ and the quadratic terms are collected, then the masses of the Higgs particles are found. This is, however, rather cumbersome. General relations for the Higgs masses in terms of the parameters are calculated in ref. [6]: The mass parameters m_{11} and m_{22} can be eliminated by minimizing the scalar potential and for the pseudoscalar and charged Higgs states one finds

$$\begin{aligned} m_{A^0}^2 &= \frac{2m_{12}^2}{\sin(2\beta)} - \frac{1}{2}v^2(2\lambda_5 + \frac{\lambda_6}{\tan\beta} + \lambda_7\tan\beta) \\ m_{H^\pm}^2 &= m_{A^0}^2 + \frac{1}{2}v^2(\lambda_5 - \lambda_4). \end{aligned} \tag{4.22}$$

The two neutral scalar states have a (non diagonal) squared mass matrix

$$\begin{aligned} \mathcal{M}_{11}^2 &= m_{A^0}^2 \sin^2\beta + v^2[\lambda_1\cos^2\beta + 2\lambda_6\sin\beta\cos\beta + \lambda_5\sin^2\beta] \\ \mathcal{M}_{22}^2 &= m_{A^0}^2 \cos^2\beta + v^2[\lambda_2\sin^2\beta + 2\lambda_7\sin\beta\cos\beta + \lambda_5\cos^2\beta] \\ \mathcal{M}_{12}^2 = \mathcal{M}_{21}^2 &= -m_{A^0}^2\sin\beta\cos\beta + v^2[(\lambda_3+\lambda_4)\sin\beta\cos\beta + \lambda_6\cos^2\beta + \lambda_7\sin^2\beta]. \end{aligned} \tag{4.23}$$

The physical mass eigenstates are obtained by rotating to a new basis

$$\begin{aligned} H^0 &= \cos\alpha\,[\sqrt{2}(\mathrm{Re}\,\phi_1^0) - v_1] + \sin\alpha\,[\sqrt{2}(\mathrm{Re}\,\phi_2^0) - v_2] \\ h^0 &= -\sin\alpha\,[\sqrt{2}(\mathrm{Re}\,\phi_1^0) - v_1] + \cos\alpha\,[\sqrt{2}(\mathrm{Re}\,\phi_2^0) - v_2], \end{aligned} \tag{4.24}$$

introducing a new mixing angle α.[5] Diagonalizing $\mathcal{M}^2$ yields

$$m_{h^0,H^0}^2 = \frac{1}{2}\left(\mathcal{M}_{11}^2 + \mathcal{M}_{22}^2 \pm \sqrt{(\mathcal{M}_{11}^2 - \mathcal{M}_{22}^2)^2 + 4(\mathcal{M}_{12}^2)^2}\right). \tag{4.25}$$

[5] In the SM the physical mass eigenstates for the Z^0 bosons and photons and are obtained by a rotation parametrized by the Weinberg angle θ_W. Compare with the footnote on page 3.

By convention $m_{h^0} \leq m_{H^0}$. To summarize, this model has six free parameters: four Higgs masses, $\tan\beta$ and α. The SM had only θ_W as a free parameter. The $W^\pm$ boson masses are derived in a manner similar to the SM: substitution of the VEV's for ϕ_1 and ϕ_2 into

$$\mathcal{D}_\mu \phi_1 \mathcal{D}^\mu \phi_1^\dagger + \mathcal{D}_\mu \phi_2 \mathcal{D}^\mu \phi_2^\dagger, \tag{4.26}$$

and collection of the quadratic terms in $W^\pm$. This yields

$$m_W^2 = \frac{1}{4}(v_1^2 + v_2^2) g^2 = \frac{1}{4} v^2 g^2. \tag{4.27}$$

For the Z^0 boson mass, recall the SM relations $\cos^2\theta_W = g^2/(g^2 + g'^2)$ and $m_Z^2 = m_W^2/(\cos^2\theta_W)$, so that

$$m_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2. \tag{4.28}$$

4.2.2 The Minimal Supersymmetric Standard Model

In terms of the two $Y = 1$ fields of the previous section one finds [6]

$$H = \begin{pmatrix} H^1 \\ H^2 \end{pmatrix} = \begin{pmatrix} \phi_1^{0*} \\ -\phi_1^- \end{pmatrix} = i\sigma^2 \phi_1^*, \quad \bar{H} = \begin{pmatrix} \bar{H}^1 \\ \bar{H}^2 \end{pmatrix} = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix} = \phi_2, \tag{4.29}$$

i. e. $< H^1 > = v_1$ and $< \bar{H}^2 > = v_2$. The scalar potential of the MSSM is

$$\begin{aligned} \mathcal{V}_{SUSY} = & m^2 (H^\dagger H + \bar{H}^\dagger \bar{H}) + \frac{1}{8} g^2 \sum_{i=1}^{3} (H^\dagger \sigma^i H + \bar{H}^\dagger \sigma^i \bar{H})^2 \\ & + \frac{1}{8} g'^2 (H^\dagger H - \bar{H}^\dagger \bar{H})^2, \end{aligned} \tag{4.30}$$

where the first contribution originates from the scalar part of the superpotential $\mathcal{W} = m(\varepsilon_{ij} H^i \bar{H}^j)$. The mass parameter is in general complex. Since it only enters the scalar potential via $|m|^2$ it is taken to be real. The second and third terms arise from the D-term contributions $D_L^i = -(g/2)(H^\dagger \sigma^i H + \bar{H}^\dagger \sigma^i \bar{H})$ and $D_Y = -(g'/2)(-H^\dagger H + \bar{H}^\dagger \bar{H})$, as specified in eq. (2.86b). The most general $SU(2)_L \otimes U(1)_Y$ invariant soft breaking terms are

$$\mathcal{V}_{soft} = \mu_1^2 H^\dagger H + \mu_2^2 \bar{H}^\dagger \bar{H} - \mu_3^2 (\varepsilon_{ij} H^i \bar{H}^j + \text{c.c.}), \tag{4.31}$$

where the mass parameters μ_1^2 and μ_2^2 are real, due to the reality of the Lagrangian. They can however be negative. Again it is possible to chose the phase of one of the Higgs fields, such that μ_3^2 is real and positive. Using

$$\sum_{i=1}^{3} (\sigma^i)_{\alpha\beta} (\sigma^i)_{\gamma\delta} = -\delta_{\alpha\beta}\delta_{\gamma\delta} + 2\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{4.32}$$

[6] Remember that H and $\bar{H}$ denote the scalar components of the Higgs superfields.

the component form of the scalar potential can be computed

$$\begin{aligned}\mathcal{V}_{MSSM} =&(m^2+\mu_1^2)[H_i^* H^i] + (m^2+\mu_2^2)[\bar{H}_i^* \bar{H}^i] - \mu_3^2\,\varepsilon_{ij}[H^i\bar{H}^j + H^{i*}\bar{H}^{j*}] \\ &+ \frac{1}{8}(g^2+g'^2)[H_i^* H^i - \bar{H}_j^*\bar{H}^j]^2 + \frac{1}{2}g^2|H_i^*\bar{H}^i|^2.\end{aligned} \tag{4.33}$$

Comparison with eq. (4.17) and noting that $\varepsilon_{ij}H^i\bar{H}^j = \phi_1^\dagger\phi_2$ and $\varepsilon_{ij}H^{i*}\bar{H}^{j*} = \phi_2^\dagger\phi_1$ gives

$$\begin{aligned}\lambda_1 = \lambda_2 = \frac{1}{4}(g^2+g'^2),\quad \lambda_3 = \frac{1}{4}(g^2-g'^2),\quad \lambda_4 = -\frac{1}{2}g^2,\quad \lambda_5=\lambda_6=\lambda_7=0 \\ m_{11}^2 = m^2+\mu_1^2,\quad m_{22}^2 = m^2+\mu_2^2,\quad m_{12}^2 = \mu_3^2.\end{aligned} \tag{4.34}$$

Thus the Higgs sector of the MSSM is CP-conserving and flavour changing neutral currents are automatically suppressed. Before Higgs mass terms can be calculated, the minimum of the scalar potential needs to be evaluated, taking the VEV's of the fields. It suffices to only consider neutral components[7] in the scalar potential, i. e.

$$\begin{aligned}\mathcal{V}^0_{MSSM} =&m^2(|H^1|^2+|\bar{H}^2|^2) + \mu_1^2|H^1|^2 + \mu_2^2|\bar{H}^2|^2 - \mu_3^2(H^1\bar{H}^2 + H^{1*}\bar{H}^{2*}) \\ &+ \frac{1}{8}(g^2+g'^2)\left(|H^1|^2 - |\bar{H}^2|^2\right).\end{aligned} \tag{4.35}$$

The (soft-breaking) parameter μ_3^2 plays a crucial role in obtaining the minimum, because without it, one always has either $< H^1 >= 0$ or $< \bar{H}^2 >= 0$. Two minimum equations are obtained by taking the partial derivative of $\mathcal{V}^0_{MSSM}$ with respect to H^1 and $\bar{H}^2$. They give further constraints and one can eliminate the parameters m^2, μ_1^2 and μ_2^2. Taking eqs. (4.22) yields

$$\begin{aligned}m_{A^0}^2 &= \frac{2m_{12}^2}{\sin(2\beta)} \\ m_{H^\pm}^2 &= m_{A^0}^2 + m_W^2\end{aligned} \tag{4.36}$$

Eq. (4.25) can be recast using m_Z and m_{A^0} as parameters:

$$m_{h^0,H^0}^2 = \frac{1}{2}\left(m_{A^0}^2 + m_Z^2 \pm \sqrt{(m_{A^0}^2+m_Z^2)^2 - 4m_Z^2 m_{A^0}^2\cos^2(2\beta)}\right). \tag{4.37}$$

Thus SUSY imposes constraints on the mass spectrum of the Higgs sector:

$$m_{H^\pm} \geq m_W, \qquad m_{H^0} \geq m_Z \tag{4.38a}$$

$$m_{h^0} \leq m_Z, \qquad m_{h^0} \leq m_{A^0}. \tag{4.38b}$$

For $m_{A^0} \to 0$ at $\tan\beta \neq 1$, $m_{h^0} > 0$. So the MSSM possesses at least one light Higgs boson.

[7] Recall that only the neutral components are taken to have non zero VEV's to ensure $U(1)_{EM}$ invariance.

Chapter 5

The Corrections

The mass relations of the last chapter are only tree-level approximations to the real physical masses. In order for the calculation to be refined, one must consider possible extensions of the theory and effects arising from radiative corrections. This will be the content of this chapter.

5.1 The Effective Lagrangian

One assumes that the MSSM is valid up to a mass scale Λ. Above this scale there are thought to be further particles and interactions. This philosophy renders the MSSM an effective low-energy theory and the phenomena — up to energies of order Λ — are described by an effective Lagrangian. Such a procedure is very general and independent of the new physics entering at the scale of Λ. One assumes, however, that no additional fields are present, so that the particle content remains that of the MSSM. The terms of the effective Lagrangian will be constructed as an expansion in powers of $1/\Lambda$:

$$\mathcal{L}_{eff} = \mathcal{L}_{MSSM} + \frac{1}{\Lambda}\mathcal{L}_1 + \frac{1}{\Lambda^2}\mathcal{L}_2 + \dots, \tag{5.1}$$

where $\mathcal{L}_{MSSM}$ is the standard MSSM Lagrangian (eq. (4.14)) of dimension four. The non renormalizable terms $\mathcal{L}_p$ are of dimension $4+p$ and can be written as

$$\mathcal{L}_p = \sum_i c_i \mathbb{O}_i^{(4+p)}, \tag{5.2}$$

where $\mathbb{O}_i^{(4+p)}$ are supersymmetric $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge invariant operators and the c_i are unknown dimensionless couplings. Thus $\mathcal{L}_{eff}$ is the most general supersymmetric gauge invariant non renormalizable Lagrangian. It will receive contributions from the Higgs superpotential, the chiral and the vector sectors and from additional soft breaking terms.

The aim of this section will be to evaluate the effect of these new terms on the mass spectrum of the MSSM Higgs sector, eqs. (4.36) and (4.37). Thus it suffices to calculate the effective Lagrangian involving only Higgs superfields.

The superpotential

The MSSM Lagrangian contains a superpotential term, as seen in eq. (4.14)

$$\mathcal{L}_{superpot} = [\mathcal{W}_{H\bar{H}}]_{\theta\theta} + [\overline{\mathcal{W}}_{H\bar{H}}]_{\bar{\theta}\bar{\theta}} = m\left[\varepsilon_{ij}H^i\bar{H}^j + \text{c.c.}\right]_{\theta\theta\bar{\theta}\bar{\theta}}, \tag{5.3}$$

(H and $\bar{H}$ are again superfields). The next higher allowed ($SU(2)_L \otimes U(1)_Y$ invariant and supersymmetric) contribution is of the form

$$\Delta_{superpot} = \left[\lambda_a(\varepsilon_{ij}H^i\bar{H}^j)^2 + \lambda_b(\varepsilon_{ij}H^i\bar{H}^j)(\varepsilon_{kl}H^{k*}\bar{H}^{l*}) + \lambda_a^*(\varepsilon_{ij}H^{i*}\bar{H}^{j*})^2\right]_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{5.4}$$

This can be easily verified, noting that the Higgs superfields are general $SU(2)_L$ doublets and thus eq. (4.17) can be referred to, yielding: λ_a-term $\sim$ λ_5-term and λ_b-term $\sim$ λ_4-term, where '$\sim$' denotes the same $SU(2)_L \otimes U(1)_Y$ properties.

The chiral sector

Recall eq. (4.12):

$$\mathcal{L}_H^{ch} = \left[H^\dagger e^{V_2}e^{-V_1}H + \bar{H}^\dagger e^{V_2}e^{V_1}\bar{H}\right]_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{5.5}$$

The next higher allowed contributions are of the form

$$\Delta_{ch} = \left[(H^\dagger e^{V_2}e^{-V_1}H)^2 + (\bar{H}^\dagger e^{V_2}e^{V_1}\bar{H})^2 + (H^\dagger e^{V_2}e^{-V_1}H)(\bar{H}^\dagger e^{V_2}e^{V_1}\bar{H})\right]_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{5.6}$$

The Vector sector

The $SU(2)_L \otimes U(1)_Y$ invariant components of the (kinetic) vector superfield Lagrangian are

$$\text{tr}[W_2^\alpha W_{2\alpha}]_{\theta\theta} + \text{tr}[\overline{W}_{2\dot{\alpha}}\overline{W}_2^{\dot{\alpha}}]_{\bar{\theta}\bar{\theta}} \quad \text{and} \quad [W_1^\alpha W_{1\alpha}]_{\theta\theta} + [\overline{W}_{1\dot{\alpha}}\overline{W}_1^{\dot{\alpha}}]_{\bar{\theta}\bar{\theta}}, \tag{5.7}$$

as specified in eq. (4.5). The most general form of these expressions is found to be (see ref. [14])

$$\begin{aligned}\Delta_{vect} =& [f_2(H)\,\text{tr}(W_2^\alpha W_{2\alpha}) + f_1(H)\,W_1^\alpha W_{1\alpha}]_{\theta\theta} \\ &+ \left[g_2(\bar{H})\,\text{tr}(\overline{W}_{2\dot{\alpha}}\overline{W}_2^{\dot{\alpha}}) + g_1(\bar{H})\,\overline{W}_{1\dot{\alpha}}\overline{W}_1^{\dot{\alpha}}\right]_{\bar{\theta}\bar{\theta}},\end{aligned} \tag{5.8}$$

where f_1, f_2, g_1, g_2 are invariant[1] analytic[2] functions of the Higgs superfields. The lowest invariant possibility allowed by eq. (5.8) is[3]

$$f_i(H) = |z_H|^2 \quad \text{and} \quad g_i(\bar{H}) = |z_{\bar{H}}|^2, \quad i = 1, 2. \tag{5.9}$$

[1] So $f_2 \to f_2$ under gauge transformations. Recalling the transformation (2.74) one finds explicitly that $f_2(H)\,\text{tr}(W_2^\alpha W_{2\alpha}) \to f_2(H)\,e^{-i\Lambda_2}\text{tr}(W_2^\alpha W_{2\alpha})e^{i\Lambda_2}$. The quantum numbers for the W and B gauge superfields are listed in tab. 4.2. Λ_2 is now a three dimensional matrix representation of $SU(2)$.

[2] $[f_i(H)]_{\bar{\theta}\bar{\theta}} = 0 = [f_i^*(H^\dagger)]_{\theta\theta}$, $[g_i(\bar{H})]_{\theta\theta} = 0 = [g_i^*(\bar{H}^\dagger)]_{\bar{\theta}\bar{\theta}}$.

[3] Check eq. (4.17).

The soft breaking terms

The soft breaking Lagrangian of eq. (4.13) for the Higgs fields reads

$$\mathcal{L}_{soft} = m_H^2 |z_H|^2 + m_{\bar{H}}^2 |z_{\bar{H}}|^2 + \mu_{H\bar{H}}^2 (z_H z_{\bar{H}} + z_H^\dagger z_{\bar{H}}^\dagger). \tag{5.10}$$

This is already the most general invariant form possible, producing no further additions to $\mathcal{L}_{eff}$.

5.1.1 The Scalar Potential

To summarize, the contributions to the scalar potential $\mathcal{V}_{MSSM}$ come from the modifications of the superpotential in eq. (5.3) and from modifications of the D-terms arising in the chiral[4] and vector sector. Their explicit form can be seen in the superfield Lagrangians of eqs. (2.71) and (2.82).

From eq. (5.4) one can construct the superpotential

$$\mathcal{W}^{corr} = \mathcal{W} + \frac{\lambda}{\Lambda}(\varepsilon_{ij} H^i \bar{H}^j)^2, \qquad \overline{\mathcal{W}}^{corr} = \overline{\mathcal{W}} + \frac{\lambda^*}{\Lambda}(\varepsilon_{ij} H^{i*} \bar{H}^{j*})^2. \tag{5.11}$$

A dimensional analysis[5] gives:

$$\dim \left[\frac{\lambda}{\Lambda}(\varepsilon_{ij} H^i \bar{H}^j)^2 \right] = \dim [\mathcal{W}] = 3. \tag{5.12}$$

The cross term in eq. (5.4) will appear directly in $\mathcal{L}_{eff}$, also multiplied with a Λ^{-1} factor. The scalar potential corresponding to these superpotential terms are

$$\begin{aligned}
\mathcal{V}^{corr}_{superpot} = & \left| \frac{\partial \mathcal{W}^{corr}}{\partial z_H^i} \right| + \left| \frac{\partial \mathcal{W}^{corr}}{\partial z_{\bar{H}}^j} \right| \\
= & |m z_{\bar{H}}^i + 2\frac{\lambda}{\Lambda}(\varepsilon_{kl} z_H^k z_{\bar{H}}^l) z_{\bar{H}}^i|^2 + |m z_H^j + 2\frac{\lambda}{\Lambda}(\varepsilon_{kl} z_H^k z_{\bar{H}}^l) z_H^j|^2 \\
= & m^2 (z_H^\dagger z_H + z_{\bar{H}}^\dagger z_{\bar{H}}) + 2m\frac{\lambda}{\Lambda}(z_H^\dagger z_H + z_{\bar{H}}^\dagger z_{\bar{H}})(\varepsilon_{kl} z_H^k z_{\bar{H}}^l + \text{c.c.}) \\
& + 4\frac{\lambda^2}{\Lambda^2}[(z_H^\dagger z_H)^2 z_{\bar{H}}^\dagger z_{\bar{H}} + z_H^\dagger z_H (z_{\bar{H}}^\dagger z_{\bar{H}})^2].
\end{aligned} \tag{5.13}$$

The first term in eq. (5.13) is the usual dimension-four operator. The next-to-leading order (NLO) correction — suppressed by Λ^{-1} — is a dimension-five operator. Finally, the Λ^{-2} term is a dimension-six operator. Thus the non renormalizable contributions are

$$\begin{aligned}
\mathcal{L}_1 &= c\,\mathbb{O}^{(5)} = 2\lambda m (z_H^\dagger z_H + z_{\bar{H}}^\dagger z_{\bar{H}})(\varepsilon_{kl} z_H^k z_{\bar{H}}^l + \text{c.c.}) \\
\mathcal{L}_2 &= d\,\mathbb{O}^{(6)} = 4\lambda^2 [(z_H^\dagger z_H)^2 z_{\bar{H}}^\dagger z_{\bar{H}} + z_H^\dagger z_H (z_{\bar{H}}^\dagger z_{\bar{H}})^2].
\end{aligned} \tag{5.14}$$

[4]There are additional corrections originating in the chiral sector, which will have an impact on the Higgs-Higgs and Higgs-boson interactions.

[5]See appendix E.

Expressing eq. (5.13) in terms of the ϕ_1 and ϕ_2 fields of the generic two-Higgs model of eq. (4.17) yields

$$\begin{aligned}\mathcal{V}^{corr}_{superpot} =& m^2(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2) + 2m\frac{\lambda}{\Lambda}(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1) \\ &+ 4\frac{\lambda^2}{\Lambda^2}[(\phi_1^\dagger\phi_1)^2\phi_2^\dagger\phi_2 + \phi_1^\dagger\phi_1(\phi_2^\dagger\phi_2)^2],\end{aligned} \tag{5.15}$$

and thus[6]

$$\lambda_6 = \lambda_7 = 2m\frac{\lambda}{\Lambda} \neq 0, \tag{5.16}$$

leading to a violation of the discrete symmetry and therefore to the appearance of flavour changing neutral currents (controlled by Λ^{-1}).

In the vector sector, the combination of eqs. (5.8) and (5.9) — keeping only the D-terms — gives

$$\frac{1}{2}(|z_H|^2 + |z_{\bar{H}}|^2)\left[(D_L^i)^2 + D_Y^2\right], \tag{5.17}$$

where D_L^i and D_Y denote the $SU(2)_L$ and $U(1)_Y$ D-terms, respectively. In order for the new terms to have the same dimension as the original ones, these corrections need to be multiplied with Λ^{-2}. The final modified form of the D-terms is obtained by joining the dimensionally corrected eq. (5.17) with the bare term in eq. (2.82), yielding

$$[D_L^iD_L^i + D_YD_Y]_{corr} = \left[1 + \frac{|z_H|^2}{\Lambda^2} + \frac{|z_{\bar{H}}|^2}{\Lambda^2}\right](D_L^iD_L^i + D_YD_Y), \tag{5.18}$$

The contribution to the D-terms arising in the chiral sector, i. e. eq. (5.6), is of the form

$$\begin{aligned}&\left[\frac{1}{2}z_H^\dagger(\widehat{D}_L + \widehat{D}_Y)z_H\right]^2, \left[\frac{1}{2}z_{\bar{H}}^\dagger(\widehat{D}_L + \widehat{D}_Y)z_{\bar{H}}\right]^2, \\ &\left[\frac{1}{2}z_H^\dagger(\widehat{D}_L + \widehat{D}_Y)z_H\right]\left[\frac{1}{2}z_{\bar{H}}^\dagger(\widehat{D}_L + \widehat{D}_Y)z_{\bar{H}}\right],\end{aligned} \tag{5.19}$$

where $\widehat{D}_L = \sum_i D_L^i(\sigma^i/2)$ and $\widehat{D}_Y = D_Y(Y/2)$, with $Y = +1$ for $z_{\bar{H}}$ and $Y = -1$ for z_H. A dimensional argument requires a suppression of the form

$$\frac{1}{\Lambda^4}. \tag{5.20}$$

Eqs. (5.18) and (5.19) are linked through the equation of motion:[7]

$$\frac{\partial \mathcal{L}_{eff}}{\partial D} = 0, \qquad D = D_L^i, D_Y. \tag{5.21}$$

[6]The dimension-five operator is actually of dimension-four term, multiplied with a mass parameter. This allows it to be compared with the renormalizable terms of eq. (4.17). This is not possible for the 'pure' dimension-six operator.

[7]Compare with eq. (2.86b).

The resulting relation will involve a correspondence of dimension-four terms (with Λ^{-2} factors) and dimension-six terms (with Λ^{-4} factors), which, when inserted into the scalar potential by virtue of eq. (2.88), will lead to even higher order operators $\mathbb{O}$. Thus, in NLO order, only the Λ^{-1} term of eq. (5.13) needs to be considered.

To sum up, the modified scalar potential is

$$\mathcal{V}_{MSSM}^{NLO} = \mathcal{V}_{SUSY} + \mathcal{V}_{soft} + 2m\frac{\lambda}{\Lambda}(z_H^\dagger z_H + z_{\bar{H}}^\dagger z_{\bar{H}})(\varepsilon_{kl} z_H^k z_{\bar{H}}^l + \text{h.c.}). \tag{5.22}$$

The matrix elements of eqs. (4.23) read

$$\begin{aligned}
\mathcal{M}_{11}^2 =& m_{A^0}^2 \sin^2\beta + m_Z^2 \cos^2\beta + 2m\frac{\lambda}{\Lambda}v^2 \sin(2\beta) \\
\mathcal{M}_{22}^2 =& m_{A^0}^2 \cos^2\beta + m_Z^2 \sin^2\beta + 2m\frac{\lambda}{\Lambda}v^2 \sin(2\beta) \\
\mathcal{M}_{12}^2 =& -\sin\beta\cos\beta(m_{A^0}^2 + m_Z^2) + 2m\frac{\lambda}{\Lambda}v^2,
\end{aligned} \tag{5.23}$$

and the Higgs masses can now be computed from eqs. (4.22) and (4.25)

$$\begin{aligned}
m_{A^0}^2 =& \frac{2}{\sin(2\beta)}\left[\mu_3^2 - m\frac{\lambda}{\Lambda}v^2\right] \\
m_{h^0,H^0}^2 =& \frac{1}{2}\left[m_{A^0}^2 + m_Z^2 + 4m\frac{\lambda}{\Lambda}v^2\sin(2\beta) \pm \sqrt{\mathbb{D}}\right] \\
\mathbb{D} :=& (m_{A^0}^2 + m_Z^2)^2 - 4m_Z^2 m_{A^0}^2 \cos^2(2\beta) \\
& - 8m\frac{\lambda}{\Lambda}v^2\sin(2\beta)(m_{A^0}^2 + m_Z^2) + 16m^2\frac{\lambda^2}{\Lambda^2}v^4.
\end{aligned} \tag{5.24}$$

The mass terms $m_{H^\pm}$, m_Z and m_W remain unchanged.

5.1.2 Evaluation

In this section a numerical analysis of the NLO corrections to the neutral Higgs masses arising in eqs. (5.24) will be presented. To this aim, a new parameter t is defined, containing all the uncertainties of the model, i. e. the mass parameter m of eqs. (4.9) or (4.31) , the fundamental mass scale Λ and the dimensionless coupling λ of eq. (5.11). One sets

$$t := m\frac{\lambda}{\Lambda}v^2, \tag{5.25}$$

where $v^2 = v_1^2 + v_2^2 = 4m_W^2/g^2$. The light neutral scalar mass, m_{h^0}, is then plotted as a function of t for special choices of $\tan\beta$ with a fixed value of m_{A^0} — see figs. 5.1. From the LEP-data the boundaries of these two parameters are known to be: $\tan\beta \geq 1$ and $m_{A^0} > 80$ [GeV]. In figs. 5.1 two limiting cases are visible, corresponding to (1.) $\tan\beta = 1$ and (2.) $\tan\beta \to \infty$:

1. At tree-level $m_{h^0}^{t.l.} = 0$. For any choice of $t > 0$ the mass of the Higgs boson arises entirely from the corrections. Eqs. (5.24) reduce (in the case

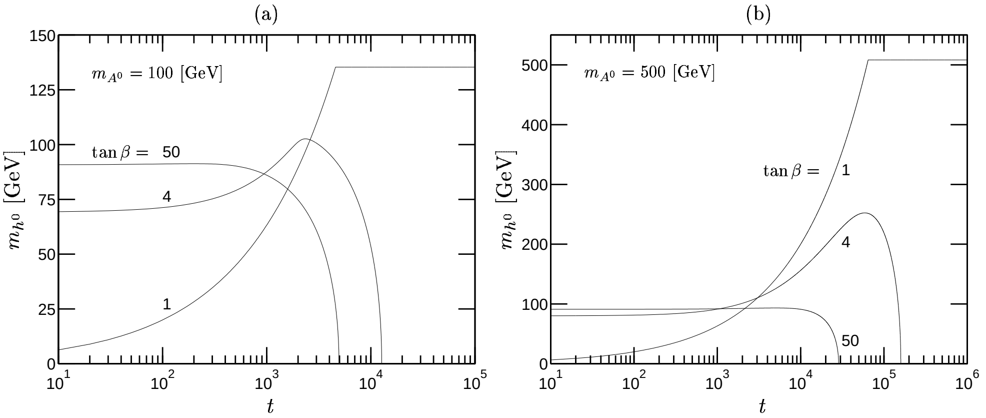

Figure 5.1: NLO-corrected Higgs mass m_{h^0} as a function of t for (a) $m_{A^0} = 100$ and (b) $m_{A^0} = 500$ [GeV]. The various curves correspond to special choices of $\tan\beta$.

of m_{h^0}) to

$$m_{h^0}^2 = \frac{1}{2}\left\{M^2 + 4t - \sqrt{M^4 - 8tM^2 + 16t^2}\right\}, \tag{5.26}$$

where $M^2 := m_{A^0}^2 + m_Z^2$. For $t \to 0$, $m_{h^0} \to m_{h^0}^{t.l.}$. The maximum value of the Higgs boson mass is reached at $t = t^{max} = M^2/4$ and is $m_{h^0}^{max} = M$. For $t > t^{max}$ the value for m_{h^0} stabilizes at the maximum $m_{h^0}^{max}$, because $4t - \sqrt{M^4 - 8tM^2 + 16t^2} = M^2$ for all $t > t^{max}$. Both tree-level bounds of eqs. (4.38b) are violated for sufficiently large values of t.

2. From eqs. (5.24) one finds

$$m_{h^0}^2 = \frac{1}{2}\left\{M^2 - \sqrt{M^4 - 4m_{A^0}^2 m_Z^2 + 16t^2}\right\}. \tag{5.27}$$

In the $t \to 0$ limit, the tree-level relation of eq. (4.37) is regained, which is also the maximum value of the Higgs mass, i. e. $m_{h^0} \leq m_{h^0}^{max} = m_{h^0}^{t.l.}$. Both tree-level bounds of eqs. (4.38b) are satisfied. In eq. (5.27) one can observe a mass-suppression due to the quadratic influence of the t-parameter and $m_{h^0} = 0$ for $t \geq 1/2\sqrt{m_{A^0}^2 m_Z^2}$.

To summarize, the NLO corrections depend strongly on the mass of the neutral pseudoscalar m_{A^0}. Although the qualitative behavior is similar for all values, the numerical outcome is strongly m_{A^0}-dependent. The maximal corrected mass is generated with a value of $\tan\beta = 1$ and is $m_{h^0}^{max} = m_{A^0}^2 + m_Z^2$. For $\tan\beta > 1$ the corrective influence becomes smaller: The mass of the neutral Higgs boson is slowly corrected — with increasing t — from the tree-level mass to a maximum value ($< m_{h^0}^{max}$) from whereon a suppression-mechanism drives it to zero, thus giving the range of the t-parameter a boundary. For $\tan\beta = \infty$, $m_{h^0} \leq m_{h^0}^{t.l.}$ for all t.

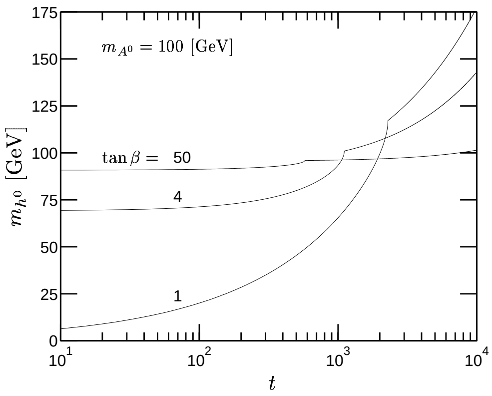

Figure 5.2: NLO-corrected Higgs mass m_{h^0} with linear corrections only.

It is interesting to note, that in the $\tan\beta \to \infty$ case, the mass-correction is solely quadratic, with no linear terms present. In fig. 5.2 the corresponding relation of eqs. (5.24) is calculated, with only the linear contributions considered. For $\tan\beta = \infty$ there is no corrective influence and the tree-level mass is unchanged. However, for any finite value of $\tan\beta$, the mass-suppression is not in effect, resulting in the unbounded growth of the neutral Higgs mass for $t \to \infty$. The case of $\tan\beta = 1$ shows, that the quadratic corrections interact with the linear contributions in a very subtle way to yield the stable plateau seen in figs. 5.1.

5.2 Radiative Corrections

One can also compute the radiative corrections to the MSSM Higgs masses. The analysis will produce all leading order logarithmic expressions for the masses. They are obtained from eqs. (4.22) and (4.23), by treating the dimensionless coupling constants $\lambda_1, \ldots, \lambda_7$ as running parameters, evaluated at the scale of electroweak symmetry breaking, $M_{ew} < M_{SUSY}$. Recall that 250 [GeV] $<$ $M_{SUSY} < 1$ [TeV]. The gauge couplings g^2 and g'^2 are known from experimental measurements at the scale of M_{ew}. If SUSY is unbroken, then eqs. (4.34) are valid:

$$\begin{aligned}
\lambda_1 &= \lambda_2 = \frac{1}{4}(g^2 + g'^2) \\
\lambda_3 &= \frac{1}{4}(g^2 - g'^2) \\
\lambda_4 &= -\frac{1}{2}g^2 \\
\lambda_5 &= 0 \\
\lambda_6 &= 0 \\
\lambda_7 &= 0.
\end{aligned} \tag{5.28}$$

Since SUSY is broken, eqs. (5.28) are regarded as boundary conditions for the running parameters, at the scale M_{SUSY}

$$\begin{aligned}
\lambda_1(M^2_{SUSY}) &= \lambda_2(M^2_{SUSY}) = \frac{1}{4}[g^2(M^2_{SUSY}) + g'^2(M^2_{SUSY})] \\
\lambda_3(M^2_{SUSY}) &= \frac{1}{4}[g^2(M^2_{SUSY}) - g'^2(M^2_{SUSY})] \\
\lambda_4(M^2_{SUSY}) &= -\frac{1}{2}g^2(M^2_{SUSY}) \\
\lambda_5(M^2_{SUSY}) &= \lambda_6(M^2_{SUSY}) = \lambda_7(M^2_{SUSY}) = 0,
\end{aligned} \tag{5.29}$$

in accordance with the tree-level MSSM relations. At scales below M_{SUSY}, the gauge and dimensionless couplings evolve according to the renormalization group equations (RGE) of the non supersymmetric two-Higgs doublet model. Their general form is

$$\frac{dp_i}{dt} = \beta_i(p_1, p_2, \dots), \qquad t := \ln\mu^2, \tag{5.30}$$

where μ is the energy scale and the p_i are the parameters of the theory, i. e. $p_i = g^2, \lambda_1, \dots$ Solving the RGE with the boundary conditions at M_{SUSY}, one can determine the λ_i at M_{ew}. The resulting values are inserted into eqs. (4.22) and (4.23). Thus the radiatively corrected Higgs masses are obtained. These corrections include the leading logarithmic adjustments summed to all orders in perturbation theory. The result is presented in ref. [6]. At leading-log order, $\lambda_5 = \lambda_6 = \lambda_7 = 0$ for all energy scales. As a consequence, the value m_{A^0} is unchanged. The charged sector will not be discussed and the emphasis is put on the calculation of the neutral Higgs masses. Their RGE-corrected parameters are

$$\begin{aligned}
\lambda_1(m_Z^2) = &+\frac{1}{4}[g^2(m_Z^2) + g'^2(m_Z^2)] + \frac{g^4(m_Z^2)}{384\pi^2\cos^4\theta_W}\left[\left(12N_c\frac{m_b^4}{m_Z^4\cos^4\beta}\right.\right. \\
&\left.-6N_c\frac{m_b^2}{m_Z^2\cos^2\beta} + P_f + P_g + P_{2H}\right)\ln\left(\frac{M^2_{SUSY}}{m_Z^2}\right) \\
&\left.+ P_t\ln\left(\frac{M^2_{SUSY}}{m_t^2}\right)\right] \\
\lambda_2(m_Z^2) = &+\frac{1}{4}[g^2(m_Z^2) + g'^2(m_Z^2)] + \frac{g^4(m_Z^2)}{384\pi^2\cos^4\theta_W}\left[\left(12N_c\frac{m_t^4}{m_Z^4\sin^4\beta}\right.\right. \\
&\left.-6N_c\frac{m_t^2}{m_Z^2\sin^2\beta} + P_t\right)\ln\left(\frac{M^2_{SUSY}}{m_t^2}\right) \\
&\left.+\left(P_f + P_g + P_{2H}\right)\ln\left(\frac{M^2_{SUSY}}{m_Z^2}\right)\right] \\
\hat\lambda_3(m_Z^2) = &-\frac{1}{4}[g^2(m_Z^2) + g'^2(m_Z^2)] - \frac{g^4(m_Z^2)}{384\pi^2\cos^4\theta_W}\left[\left(-3N_c\frac{m_b^2}{m_Z^2\cos^2\beta}\right.\right. \\
&\left.+P_f + P'_g + P'_{2H}\right)\ln\left(\frac{M^2_{SUSY}}{m_Z^2}\right) \\
&\left.+\left(-3N_c\frac{m_t^2}{m_Z^2\sin^2\beta} + P_t\right)\ln\left(\frac{M^2_{SUSY}}{m_t^2}\right)\right].
\end{aligned} \tag{5.31}$$

In eqs. (5.31) the newly defined terms are: $\hat{\lambda}_3 := \lambda_3 + \lambda_4$, N_c and N_g are respectively the number of colours and generations, i. e. $N_c = N_g = 3$ and the top and bottom quark masses are denoted by m_t^2 and m_b^2. Further definitions are

$$\begin{aligned} P_t :=& \quad N_c(1 - 4e_u \sin^2\theta_W + 8e_u^2 \sin^4\theta_W) \\ P_f :=& \quad N_g\{N_c[2 - 4\sin^2\theta_W + 8(d_d^2 + e_u^2)\sin^4\theta_W] \\ & + [2 - 4\sin^2\theta_W + 8\sin^4\theta_W]\} - P_t \\ P_g :=& - 44 + 106\sin^2\theta_W - 62\sin^4\theta_W \\ P_g' :=& \quad 10 + 34\sin^2\theta_W - 26\sin^4\theta_W \\ P_{2H} :=& - 10 + 2\sin^2\theta_W - 2\sin^4\theta_W \\ P_{2H}' :=& \quad 8 - 22\sin^2\theta_W + 10\sin^4\theta_W, \end{aligned} \tag{5.32}$$

where $e_u = 2/3$ and $e_d = -1/3$ give the up and down quark charges. The subscripts t, f, g and $2H$ indicate contributions from the top quark, the fermions (without the t-quark), the gauge bosons and the two Higgs doublets. With these corrected parameters the neutral Higgs masses can be computed. For ease of calculation only the leading $\mathcal{O}(m_t^4)$ terms will be considered, i. e.

$$\begin{aligned} \lambda_1(m_Z^2) =& \quad \frac{1}{4}[g^2(m_Z^2) + g'^2(m_Z^2)] \\ \lambda_2(m_Z^2) =& \quad \frac{1}{4}[g^2(m_Z^2) + g'^2(m_Z^2)] \\ & + \frac{g^4(m_Z^2)}{384\pi^2\cos^4\theta_W}\left(12N_c\frac{m_t^4}{m_Z^4\sin^4\beta}\right)\ln\left(\frac{M_{SUSY}^2}{m_t^2}\right) \\ \hat{\lambda}_3(m_Z^2) =& - \frac{1}{4}[g^2(m_Z^2) + g'^2(m_Z^2)], \end{aligned} \tag{5.33}$$

The matrix elements are then

$$\begin{aligned} \mathcal{M}_{11}^2 =& m_{A^0}^2 \sin^2\beta + m_Z^2\cos^2\beta \\ \mathcal{M}_{22}^2 =& m_{A^0}^2 \cos^2\beta + m_Z^2\sin^2\beta + \frac{g^2 m_Z^2 \sin^2\beta}{96\pi^2\cos^2\theta_W}\left(12N_c\frac{m_t^4}{m_Z^4\sin^4\beta}\right)\ln\left(\frac{M_{SUSY}^2}{m_t^2}\right) \\ \mathcal{M}_{12}^2 =& - \sin\beta\cos\beta(m_{A^0}^2 + m_Z^2), \end{aligned} \tag{5.34}$$

giving a neutral Higgs mass of

$$m_{h^0}^2 = \frac{1}{2}\left\{M_+^2 + \Delta - \sqrt{M_+^4 - 4m_Z^2 m_{A^0}^2\cos^2(2\beta) + 2\Delta\cos(2\beta)M_-^2 + \Delta^2}\right\}, \tag{5.35}$$

where $M_+^2 := m_{A^0}^2 + m_Z^2$, $M_-^2 := m_{A^0}^2 - m_Z^2$ and

$$\Delta := \frac{g^2 m_Z^2 \sin^2\beta}{96\pi^2\cos^2\theta_W}\left(12N_c\frac{m_t^4}{m_Z^4\sin^4\beta}\right)\ln\left(\frac{M_{SUSY}^2}{m_t^2}\right). \tag{5.36}$$

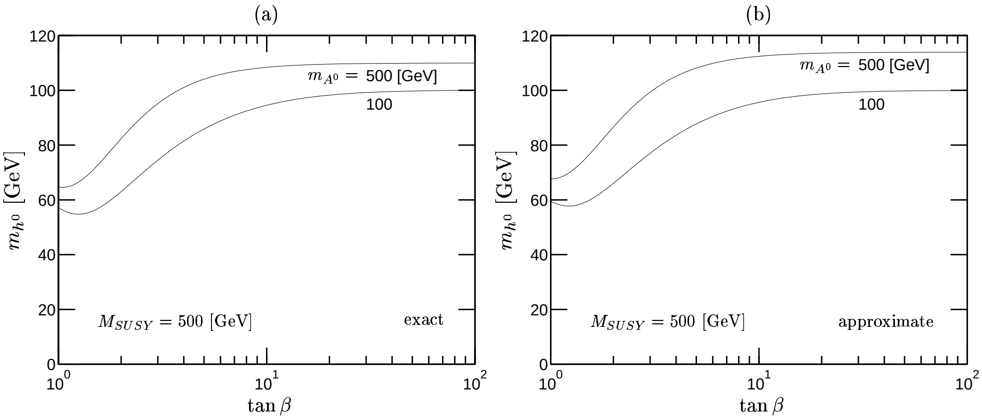

Figure 5.3: Comparison of the RGE-corrected Higgs masses (a) due to the exact equations and (b) due to the equations of order $\mathcal{O}(m_t^4)$.

In figs. 5.3 the neutral Higgs masses are calculated using the exact relations arising from eqs. (5.31) and the approximated relations from eqs. (5.33), i. e. eq. (5.35). The accuracy is hardly decreased, justifying the approximation. For larger values of m_{A^0} the difference becomes more noticeable. In the limit of $\tan\beta \to \infty$ the value of m_{h^0} reaches a maximum of

$$m_{h^0}^2 = \frac{1}{2}\left\{M_+^2 + \Delta - \sqrt{M_+^4 + \Delta^2}\right\}, \tag{5.37}$$

as seen in fig 5.3 (b). The minimum of the Higgs mass is obtained for values of $\tan\beta = 1$ and $m_{A^0} \geq 300$ [GeV]. The tree-level bounds of eqs. (4.38b) are both violated for large values of m_{A^0} and $\tan\beta$. If m_{A^0} is taken to be less or equal than 100 [GeV] then m_{h^0} converges to this value of m_{A^0} in the large $\tan\beta$ limit, and thus the tree-level bound $m_{h^0} \leq m_{A^0}$ is satisfied.

In figs. 5.4 the dependency on the breaking parameter M_{SUSY} is shown. In the case of heavy Higgs pseudoscalars ($m_{A^0} > 100$ [GeV]) the corrections

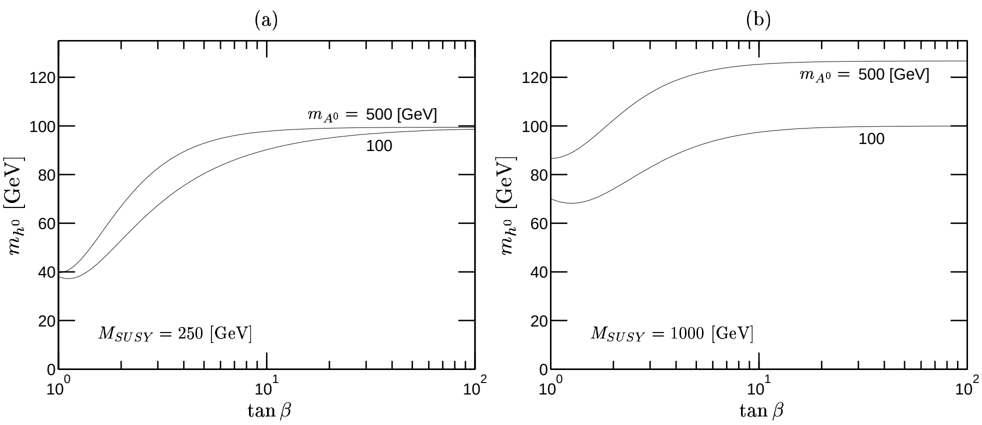

Figure 5.4: RGE-correcteed Higgs masses for different values of M_{SUSY}.

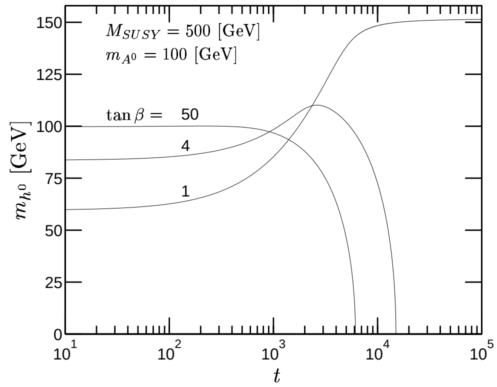

Figure 5.5: The RGE and NLO corrections are simultaneously calculated for a NLO-value of $m_{A^0} = 100$ [GeV] and RGE-value of $M_{SUSY} = 500$ [GeV].

are amplified with increasing M_{SUSY}. For $m_{A^0} = 100$ [GeV] one only finds the minimum of m_{h^0} noticeably upwards shifted.

5.3 The Total Corrections

What happens when NLO and RGE corrections are simultaneously considered? In fig. 5.5 the adjusted light neutral Higgs masses are plotted with both types of corrections. The mass matrix elements are obtained from the combination of eqs. (5.23) and (5.34), yielding

$$\begin{aligned}\mathcal{M}_{11}^2 =& m_{A^0}^2\sin^2\beta + m_Z^2\cos^2\beta + 2m\frac{\lambda}{\Lambda}v^2\sin(2\beta)\\ \mathcal{M}_{22}^2 =& m_{A^0}^2\cos^2\beta + m_Z^2\sin^2\beta + 2m\frac{\lambda}{\Lambda}v^2\sin(2\beta)\\ &+\frac{g^2m_Z^2\sin^2\beta}{96\pi^2\cos^2\theta_W}\left(12N_c\frac{m_t^4}{m_Z^4\sin^4\beta}\right)\ln\left(\frac{M_{SUSY}^2}{m_t^2}\right)\\ \mathcal{M}_{12}^2 =& -\sin\beta\cos\beta(m_{A^0}^2+m_Z^2) + 2m\frac{\lambda}{\Lambda}v^2.\end{aligned} \tag{5.38}$$

Introducing abbreviations of the form

$$\begin{aligned} t :=& m\frac{\lambda}{\Lambda}v^2\\ M_+^2 :=& m_{A^0}^2 + m_Z^2\\ M_-^2 :=& m_{A^0}^2 - m_Z^2\\ \Delta :=& \frac{g^2m_Z^2\sin^2\beta}{96\pi^2\cos^2\theta_W}\left(12N_c\frac{m_t^4}{m_Z^4\sin^4\beta}\right)\ln\left(\frac{M_{SUSY}^2}{m_t^2}\right),\end{aligned} \tag{5.39}$$

the mass of the neutral Higgs boson is found to be

$$\begin{aligned} m_{h^0}^2 =& \frac{1}{2}\left\{M_+^2 + 4t\sin(2\beta) + \Delta - \sqrt{\mathbb{D}}\right\} \\ \mathbb{D} :=& M_+^4 - 4m_Z^2 m_{A^0}^2 \cos^2(2\beta) \\ & - 8t\sin(2\beta)M_-^2 + 16t^2 + 2\Delta\cos(2\beta)M_-^2 + \Delta^2. \end{aligned} \tag{5.40}$$

Fig. 5.5 shows the characteristic behavior due to the NLO corrections as discussed in section 5.1.2, but the numerical values are influenced by the RGE corrections:

1. $\tan\beta = 1$:

$$m_{h^0} = \begin{cases} \sqrt{M_+^2 + \frac{\Delta}{2}}, & \text{for } t \text{ large.} \\ \sqrt{\frac{1}{2}\left[M_+^2 + \Delta - \sqrt{M_+^4 + \Delta^2}\right]}, & \text{for } t = 0. \end{cases} \tag{5.41}$$

2. $\tan\beta = \infty$:

$$m_{h^0} \leq \sqrt{\frac{1}{2}\left[M_+^2 + \Delta - \sqrt{M_+^4 - 4m_{A^0}^2 m_Z^2 - 2\Delta M_-^2 + \Delta^2}\right]}. \tag{5.42}$$

In fig. 5.6 the tree-level Higgs mass values are numerically subtracted from the corrected masses of fig. 5.5, i.e. $m_{h^0}^{tot} - m_{h^0}^{t.l.}$, giving the raw value of the corrections.

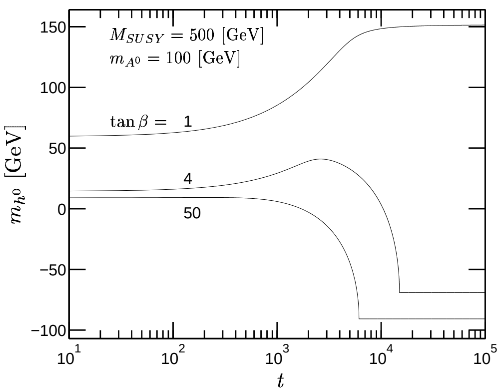

Figure 5.6: The RGE and NLO corrections are simultaneously calculated for a NLO-value of $m_{A^0} = 100$ [GeV] and RGE-value of $M_{SUSY} = 500$ [GeV], but the tree-level Higgs masses are numerically subtracted, yielding the true value of the corrections.

Chapter 6

Summary and Conclusions

The SM is a well tested and well established framework in which all the non-gravitational forces are united[1]. Soon it was realized that in spite of the success and the accuracy of the theory it must be extended to address some defects by which it is plagued. For a brief review of the questions raised by the SM see reference [15].

In this paper the problem of the origin of mass is discussed, which in the SM is thought to come from a fundamental scalar Higgs field[2]. The search for this elusive particle is also of paramount experimental importance: By 2010 a new accelerator will be fully operational at CERN, called the Large Hadron Collider (LHC). It should be powerful enough to detect the Higgs particle. However, if the Higgs boson is not detected it is still possible to take the following point of view: The modifications to the Higgs mass calculated in this paper due to SUSY correct the mass out of the detection range. Unfortunately the accuracy of the calculation was restricted to next-to-leading order which leaves an inconclusive result due to the wealth of free parameters. Analyzing the next higher corrections could give a clearer answer, but the mathematical involvement is quite arduous.

Incidently, the LHC will also be searching for superpartner particles, which are not only an essential ingredient to the extension of the SM via SUSY, but play a fundamental role in string/M-theory[3]. So a lot of expectations and apprehensions await the LHC.

[1] For a brief note on the unification of all forces, see appendix F.

[2] Within the framework of string/M-theory it seems conceivable that mass terms originate solely from the compactification of the 11-dimensional space-time.

[3] For a remark on SUSY in string theory, see footnote appendix F.

Appendix A

Notation and Conventions

- $\hbar = c = \varepsilon_0 = 1$.
- The Einstein summation convention is assumed if the summation is not explicitly shown.
- † denotes the hermitian conjugate, * the complex conjugate and ^{t} the transpose of a value.
- Greek indices from the middle of the alphabet $(\mu, \nu, \dots)$ range from 0 to 3. The indices $\alpha, \beta, \dots$ are usually used to label two-component objects. Occasionally they are taken to range from 1 to 4, with specific declaration. Dotted indices $(\dot{\alpha}, \dot{\beta}, \dots)$ always have values 1 or 2. They are introduced in appendix D. Latin indices (i, j, ...) range from 1 to 3.
- For a matrices A_{ij} the index placement used in the summation convention is

$$(A^t)_i{}^j = A_j{}^i = A^j{}_i, \tag{A.1}$$

i. e. for a column vector $\vec{w} = A\vec{v}$ one finds

$$w_i = A_i{}^j v_j, \tag{A.2}$$

and for $\vec{w}^{\,t} = \vec{v}^{\,t} A^t$

$$w_i = v_j A^j{}_i. \tag{A.3}$$

- The metric for flat space-time is

$$\eta^{\mu\nu} = \mathrm{diag}\,(+, -, -, -). \tag{A.4}$$

- The totally antisymmetric Levi-Civita tensor is denoted by ε (i. e. $\varepsilon_{\alpha\beta}$, ε_{ijk} and $\varepsilon_{\mu\nu\gamma\rho}$).
- The 4×4 gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{A.5}$$

In the *standard* representation they are taken to be

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \vec{\gamma} = \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \quad \text{(A.6)}$$

The *chiral* (or *Weyl*) representation is

$$\gamma^0 = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \quad \text{(A.7)}$$

For the *Dirac* representation one finds

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \quad \text{(A.8)}$$

The Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{(A.9)}$$

have been used.

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, \qquad [\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k. \quad \text{(A.10)}$$

- The charge conjugation matrix C is defined as

$$C = i\gamma^2\gamma^0, \quad \text{(A.11)}$$

with

$$C^2 = I_4, \qquad C = -C^\dagger = C^t, \quad \text{(A.12)}$$

and

$$\begin{aligned} C^{\text{stand}} &= C^{\text{Weyl}} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \\ C^{\text{Dirac}} &= C^{\text{Majorana}} = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}. \end{aligned} \quad \text{(A.13)}$$

- Partial derivatives in space-time coordinates are

$$\begin{aligned} \partial_\mu &= \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{\partial}{\partial t} \\ \nabla \end{pmatrix} \\ \partial^\mu &= \frac{\partial}{\partial x_\mu} = \begin{pmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{pmatrix}, \end{aligned} \quad \text{(A.14)}$$

and in anticommuting coordinates

$$\begin{aligned} \partial_\alpha &= \frac{\partial}{\partial \theta^\alpha}, \qquad \partial^\alpha = \frac{\partial}{\partial \theta_\alpha} \\ \bar{\partial}_{\dot{\alpha}} &= \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}, \qquad \bar{\partial}^{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}. \end{aligned} \quad \text{(A.15)}$$

Appendix B

Elements of Group Theory

In the literature there is a certain ambiguity to the meaning of the term 'representation':

1. In the rigorous mathematical definition, a linear representation is a specific realization of the multiplication of the group elements by matrices, i. e. a mapping of the abstract group elements to a set of matrices. For brevity the representation is identified with the matrices themselves. The set of basis vectors spanning the vector space on which the matrices act as linear transformations form a *multiplet*.

2. An alternative approach[1] uses the transformation properties of a given set of (physical) fields under the elements of a specific group to construct a representation of the considered group:

$$\phi'^i(x) = U(g)\phi^i(x)U^{-1}(g) = \mathcal{R}^i{}_j(-g)\phi^j(x'). \tag{B.1}$$

   In this equation the $\phi^i(x)$ $(i = 1, \ldots, N)$ denote a set of field operators on the vector space of physical states (2^{nd} quantization) transforming by virtue of $U(g)$, which is a representation of the group element g taken as a linear transformation on the same space. The group elements also induces a transformation in the physical x-space, thus defining x'. The matrix $\mathcal{R}(g)$ is a N-dimensional representation of the group. Eq. (B.1) specifies the transformation properties of a vector field. It can be easily modified for scalar or tensor fields, by appropriate choice of the representation matrix $\mathcal{R}$. So the fields span the representation space and $\big(\phi^1(x), \ldots, \phi^N(x)\big)$ is a multiplet.

B.1 External Symmetries

The symmetry groups of four-dimensional space-time are the (proper) Lorentz group $\mathscr{L}$ and the Poincaré group $\mathscr{P}$, the former consisting of 'rotations' in space-time (i. e. three-dimensional rotations and Lorentz boosts) and the latter of rotations and translations in space-time (i. e. $\mathscr{L}$ extended by the inclusion

[1] This is often seen in the physics literature.

of four-dimensional translations). The physical significance of these groups is that

- particle states transform as unitary representations of $\mathscr{P}$,
- fields transform as finite dimensional representations of $\mathscr{L}$.

B.1.1 The Poincaré Algebra

The Lie algebra of the Poincaré group generators is

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}) \tag{B.2a}$$
$$[M^{\mu\nu}, P^{\rho}] = -i(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu}) \tag{B.2b}$$
$$[P^{\mu}, P^{\nu}] = 0. \tag{B.2c}$$

$P^\mu = i\partial^\mu$ generate four-dimensional translations and $M^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ generate Lorentz transformations, thus eq. (B.2a) is the Lie algebra of the Lorentz group. Eq. (B.2b) can be written as $[M^{\mu\nu}, P^\rho] = (V^{\mu\nu})_\lambda P^\lambda$, implying that P^μ transforms as a vector under Lorentz transformations.

B.1.2 Representations of the Lorentz Group

A general group element of $\mathscr{L}$ induces the coordinate transformation

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{B.3}$$

where the Λ matrices satisfy

$$\Lambda^{-1} = \eta\Lambda^t\eta^{-1}, \qquad \det\Lambda = 1, \qquad \Lambda^0_0 \geq 1. \tag{B.4}$$

Thus the mathematical designation of $\mathscr{L}$ is $SO(1,3)$. Infinitesimal Lorentz transformations (parametrized by ω) can be written as

$$\Lambda^{\kappa\lambda}(\omega) = \eta^{\kappa\lambda} + \frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^{\kappa\lambda} + \mathcal{O}(\omega^2). \tag{B.5}$$

It is possible to decompose an arbitrary Lorentz transformation into a product of a three-dimensional rotation and a boost. We can then identify

$$J^i := \sum_k \varepsilon^{ijk}M^{jk} \quad \text{and} \quad K^i := M^{0i}, \tag{B.6}$$

where $\vec{J}$ and $\vec{K}$ generate the rotations and the boosts respectively. The Lie algebra expressed in terms of these new generators reads

$$[J^m, J^n] = i\sum_k \varepsilon^{mnl}J^l \tag{B.7a}$$
$$[K^m, J^n] = i\sum_k \varepsilon^{mnl}K^l \tag{B.7b}$$
$$[K^m, K^n] = -i\sum_k \varepsilon^{mnl}J^l, \tag{B.7c}$$

where eq. (B.7a) is the $SO(3)$ Lie algebra as expected. One can introduce yet another basis in the space of the generators, taken to be

$$M^i := \frac{1}{2}(J^i + iK^i), \qquad N^i := \frac{1}{2}(J^i - iK^i). \tag{B.8}$$

These new generators satisfy

$$\begin{aligned} [M^i, M^j] &= i \sum_k \varepsilon^{ijk} M^k \\ [M^i, N^j] &= 0 \\ [N^i, N^j] &= i \sum_k \varepsilon^{ijk} N^k, \end{aligned} \tag{B.9}$$

which is the Lie algebra of the direct product of the two complex $SU(2)$ algebras, generated by $\vec{M}$ and $\vec{N}$ respectively. This shows that the Lie algebra of the Lorentz group $SO(1,3)$ is equivalent to the Lie algebra of the group $SU(2)_M \otimes SU(2)_N$.[2] Therefore the irreducible finite dimensional representations of the Lorentz group can be characterized by the total spin of $SU(2)_M$ and $SU(2)_N$, denoted by (m, n). The representations $(\frac{1}{2}, 0)$ — corresponding to $\vec{J} = \vec{\sigma}/2$, $\vec{K} = -i\vec{\sigma}/2$ — and $(0, \frac{1}{2})$ — corresponding to $\vec{J} = \vec{\sigma}/2$, $\vec{K} = i\vec{\sigma}/2$ — are called the *fundamental representations* of the Lorentz group. They are inequivalent, irreducible and two-dimensional. The basis vectors in the representation space are called (two-component) *spinors*[3]. If one extends the Lorentz group by parity transformations, then the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations interchange and the two spinors are combined to form a single *Dirac* or four-component spinor[4]. They form a basis for the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, which is now irreducible and four-dimensional. The transformation properties of the physical fields under the Lorentz group are:

- Scalar fields (spin-0) transform as $(0, 0)$.
- Spinor fields (spin-$\frac{1}{2}$) transform as $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$.
- Vector fields (spin-1) transform as $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$, the product of two spinors.
- Spin-$\frac{3}{2}$ fields transform as $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, 0)$, the product of a vector and a spinor.
- Tensor fields (spin-2) transform as $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})$, the product of two vectors.

Considering Lorentz transformations (parametrized by ω) eq. (B.1) takes the generic form

$$\Phi(x) \to \Phi'(x) = \mathcal{R}(-\omega)\, \Phi(x') = \mathcal{R}(-\omega)\, \Phi\left\{\Lambda(\omega)x\right\}. \tag{B.10}$$

[2] It should be noted that $SU(2)_M \otimes SU(2)_N$ is a compact group in contrast to $SO(1,3)$ and that the generators M^i and N^i are not hermitian.

[3] See appendix D.

[4] See appendix C.

The representation matrix $\mathcal{R}$ is determined by the nature (i. e. transformation property) of the field denoted by Φ:

$$\begin{aligned} \mathcal{R} &\to 1, \text{ for a scalar field } \phi(x), \\ \mathcal{R} \to S_{\alpha\beta}(-\omega) = \delta_{\alpha\beta} - \frac{i}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})_{\alpha\beta} + \mathcal{O}(\omega^2), &\text{ for a Dirac spinor field } \psi_\alpha(x), \\ \mathcal{R} \to \Lambda_{\mu\nu}(-\omega) = \eta^{\mu\nu} - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})_{\mu\nu} + \mathcal{O}(\omega^2), &\text{ for a vector field } V_\mu(x), \end{aligned}$$

or in a compact infinitesimal notation

$$\delta^{Lorentz}\Phi = -\frac{i}{2}\omega R\Phi, \tag{B.12}$$

where $R = (1, \Sigma, J)$ satisfies the Lie algebra of the Lorentz group $SO(1,3)$, i. e. eq. (B.2a). For spinors the corresponding generators are

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \tag{B.13}$$

where the γ^μ matrices are given in appendix A, and for vectors one finds

$$(J^{\rho\sigma})^{\mu\nu} = i(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}). \tag{B.14}$$

A more specific way to write eq. (B.10) is:

$$\begin{aligned} \phi'(x) &= \phi(\Lambda x) \\ \psi'^\alpha(x) &= \left[e^{-i\omega\Sigma}\right]^\alpha{}_\beta \psi^\beta(\Lambda x) \\ V'^\mu(x) &= \left[e^{-i\omega J}\right]^\mu{}_\nu V^\nu(\Lambda x). \end{aligned} \tag{B.15}$$

B.1.3 Rotations

Rotations in N-dimensional space are given by the group $SO(N)$:

- The dimension is $\frac{1}{2}N(N-1)$ and the rank[5] is $\frac{N}{2}$ for N even or $\frac{N-1}{2}$ for N odd.
- $x^i \to x'^i = O^i{}_j x^j$ for $O \in SO(N)$.
- $O = O(\theta) = \exp\left(i\sum_{a=1}^{N(N-1)/2}\theta^a t^a\right)$. The rotation angles θ are the parameters of the transformation.
- The representation defined via the transformation properties of N (scalar or vector) fields $\phi(x)$ is generated by

  $$(M^{ij})_{kl} = i(\delta^j_k\delta^i_l - \delta^i_k\delta^j_l), \tag{B.16}$$

  and the commutation relations are

  $$[M^{ij}, M^{lm}] = -i(\delta^{il}M^{jm} - \delta^{jl}M^{im} - \delta^{im}M^{jl} + \delta^{mj}M^{il}). \tag{B.17}$$

  Setting $U(\omega) = e^{i\omega_{ij}M^{ij}}$ one finds

[5] The number of generators that can be simultaneously diagonal.

– for scalar fields: $U(\omega)\phi(x)U^{-1}(\omega) = \phi(x')$.
– for vector fields: $U(\omega)\phi^i(x)U^{-1}(\omega) = \left[O(-\theta)\right]^i{}_j\,\phi^j(x')$.

- The spinor representation defined via the transformation properties of N spinor fields is generated by

$$M^{ij} = \frac{i}{4}[\Gamma^i, \Gamma^j], \tag{B.18}$$

where M^{ij} satisfy eq. (B.17) and the vectors $\vec{\Gamma}$ satisfy the Clifford algebra

$$\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}. \tag{B.19}$$

The spinor representation of $SO(4)$ — respectively $SO(1,3)$ — is constructed using the gamma matrices of appendix A, see eqs. (B.13) and (A.5).

B.2 Internal Symmetries

The advent of *gauge theories*[6] in physics brought into consideration a new type of symmetry: local gauge symmetry. In a first step the Lagrangian $\mathcal{L}$ is taken to be invariant under some global gauge group G, i. e. under the infinitesimal transformation

$$\phi^i(x) \to \phi^{i\prime}(x) = \phi^i(x) + \delta\phi^i(x); \qquad i = 1, \ldots, N, \tag{B.20}$$

where

$$\delta\phi^i(x) = i\omega_a(t^a)^i{}_j\phi^j(x). \tag{B.21}$$

The t^a ($a = 1, \ldots, \dim G$) are the hermitian generators of the group G and satisfy the Lie algebra

$$[t^a, t^b] = i\sum_c f^{abc}t^c, \tag{B.22}$$

with f^{abc} as the structure constants of the group. It is always possible to define a matrix representation of the generators

$$(t^a)^{bc} = -if^{abc}, \tag{B.23}$$

which satisfies eq. (B.22). This is called the *adjoint representation*. The normalization is taken to be $\mathrm{tr}(t^a t^b) = \kappa\delta^{ab}$. This fixes the normalization for all

[6]The geometrical structure of the theory is that of a *fiber bundle*, in which an 'internal' space is erected at every point in space-time. This means, that a wave function $\phi(x)$ has a definite direction in the internal space at the point x. The problem of correlating the relative directions of ϕ at different space-time points is closely related to the affine connection of general relativity, which allows the definition of parallel transport. In gauge theory one introduces a (vector) gauge field, which generates infinitesimal rotations in the internal space as a result of an external space-time displacement.

other representations.[7] The f^{abc} are then totally antisymmetric. Eq. (B.21) corresponds to

$$\phi'^i(x) = \left[e^{i\omega_a t^a}\right]^i_j \phi^j(x). \tag{B.24}$$

The transformation properties of the fields ϕ^i specify a (N-dimensional) representation of the gauge group, which also determines the matrix representation for the t^a. The symmetry is turned into a local one (the symmetry is 'gauged'), by replacing ω with $\omega(x)$, i.e. the transformation parameters become space-time dependent. To retain the invariance of $\mathcal{L}$, a *gauge-covariant derivative* is introduced

$$\partial_\mu \to D_\mu := \partial_\mu + ig \sum_a t^a V^a_\mu, \tag{B.25}$$

which contains a set of (vector) gauge fields V^a_μ and a coupling constant g associated with the group G. The gauge fields form a multiplet belonging to the adjoint representation of G.

$SU(N)$ characterizes the gauge groups encountered in physics:

- The dimension is $N^2 - 1$ and the rank $N - 1$.
- $U \in SU(N) \Rightarrow U = \exp\left(i \sum_{a=1}^{N^2-1} \theta^a t^a\right)$. One finds $UU^\dagger = U^\dagger U = I_N$.
- The representation defined via the transformation properties of N vectors $\psi^i = (\psi^1, \ldots, \psi^N)$:

$$\psi^i \to \psi'^i = U^i_j \psi^j. \tag{B.26}$$

  It is denoted as $\mathbf{N}$.

- The conjugate representation $\bar{\mathbf{N}}$ is defined via the transformation properties of the N vectors ψ^*_i:

$$\psi^*_i \to \psi'^*_i = \psi^*_j (U^\dagger)^j_i. \tag{B.27}$$

- $\mathbf{N} \otimes \bar{\mathbf{N}} = (\mathbf{N^2 - 1}) \oplus \mathbf{1}$.

One finds that $SO(2) \sim U(1)$, $SO(3) \sim SU(2)$ and $SO(4) \sim SU(2) \otimes SU(2)$.

[7] For $SU(N)$ $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$, $SO(N)$ $(N \geq 3)$ $\mathrm{tr}(t^a t^b) = \delta^{ab}$. $SO(3)$ is normalized as $SU(2)$.

Appendix C

Four-Component Spinors

Recall from appendix B that the Dirac spinors form a basis for a representation of the Lorentz group. The generators of $\mathscr{L}$, $M^{\mu\nu}$, obey the commutation relations of eq. (B.2a). A matrix representation of the generators is given by

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{C.1}$$

The corresponding (Lorentz) group element is $S = e^{-i\omega_{\mu\nu}\Sigma^{\mu\nu}}$ and acts on the basis space of Dirac four-spinors

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \tag{C.2}$$

i. e.

$$\psi'(x) = S\psi(x') = S\,\psi\,[\Lambda(\omega)x]\,, \tag{C.3}$$

or

$$\delta^{Lotentz}\psi = -\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\psi. \tag{C.4}$$

The adjoint spinor $\bar{\psi} = \psi^\dagger\gamma^0$ transforms as

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}S^{-1}. \tag{C.5}$$

It is possible to construct a quantity related to the adjoint spinor that has the same transformation properties as ψ. It is defined via the charge conjugation matrix and is of the form

$$\psi^c := C\bar{\psi}^t, \tag{C.6}$$

which indeed transforms as $\psi^{c\prime} = S\psi^c$. Depending on the choice of the γ matrices one attains different characteristics for the spinors.

- In the Dirac representation one gets the Dirac spinor, i. e. a complex-valued field with four components describing a massive spin-$\frac{1}{2}$ field.

- In the Weyl and the standard representation one can define *chiral projectors*

$$P_L = \frac{1}{2}(I_4 + \gamma^5) = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \frac{1}{2}(I_4 - \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}, \quad \text{(C.7)}$$

with $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L P_R = P_R P_L = 0$. The Dirac spinor takes the form

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \text{(C.8)}$$

where ψ_L and ψ_R are *Weyl spinors* and are subject to the constraints

$$P_R \psi_L = 0$$
$$P_L \psi_R = 0.$$

They describe massless spin-$\frac{1}{2}$ particles. Again this illustrates the fact, that Dirac spinors span a reducible representation of the Lorentz group, which decomposes into the irreducible inequivalent representation, spanned by the Weyl spinors (compare with section B.1.2).

- A *Majorana spinor* is define by the condition $\psi = \psi^c$. Using the standard representation this can be written as

$$\psi = \begin{pmatrix} \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix}. \quad \text{(C.9)}$$

Appendix D

Two-Component Spinors

In appendix C the Dirac spinor ψ was decomposed into two-component Weyl spinors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} . \tag{D.1}$$

The left and right-handed Weyl spinors span two inequivalent irreducible representations of the Lorentz group (section B.1.2). They transform under the Lorentz group via matrices M and M^* from $SL(2,\mathbb{C})$. This illustrates the fact, that there is a natural correspondence between the groups $SO(1,3)$ and $SL(2,\mathbb{C})$. The connection is made by associating with each space-time point x^μ a 2×2 hermitian matrix X:

$$x^\mu \longleftrightarrow X = \sigma_\mu \cdot x^\mu , \tag{D.2}$$

where $\sigma_i, i = 1,2,3$, are the Pauli matrices and σ_0 is the identity matrix. A Lorentz transformation Λ (on the four-vector x^μ) induces a linear transformation on the matrix X, preserving its hermiticity

$$x'^\mu = \Lambda^\mu_\nu x^\nu \longleftrightarrow X' = M X M^\dagger . \tag{D.3}$$

The matrices M satisfy

$$M \varepsilon M^t = \varepsilon \qquad \Longleftrightarrow \qquad \det M = 1, \tag{D.4}$$

i.e. $M \in SL(2,\mathbb{C})$.

A two-component spinor ψ transforms under Lorentz transformations as

$$\psi'_\alpha = M_\alpha^{\ \beta} \psi_\beta . \tag{D.5}$$

This corresponds to the irreducible representation labelled by $(0, \frac{1}{2})$. By convention spinors transforming in accordance with the $(\frac{1}{2}, 0)$ representation are written with *dotted* indices and a *bar*:[1]

$$\bar{\psi}'_{\dot\alpha} = (M^*)_{\dot\alpha}^{\ \dot\beta} \bar{\psi}_{\dot\beta} = \psi_{\dot\beta} (M^\dagger)^{\dot\beta}_{\ \dot\alpha} . \tag{D.6}$$

[1] The convention for the placement of the indices is given in appendix A.

Using the ε matrix[2] it is possible to introduce the contravariant components of the two-spinors, i. e.

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \bar\psi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \tag{D.7}$$

with the transformation properties

$$\begin{aligned}\psi'^\alpha &= (M^{t-1})^\alpha{}_\beta\psi^\beta = \psi^\beta(M^{-1})_\beta{}^\alpha\\ \bar\psi'^{\dot\alpha} &= (M^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}\bar\psi^{\dot\beta} = \bar\psi^{\dot\beta}(M^{*-1})_{\dot\beta}{}^{\dot\alpha}.\end{aligned} \tag{D.8}$$

One defines the (Lorentz) scalar quantities

$$\begin{aligned}\eta\chi &:= \eta^\alpha\chi_\alpha = -\eta_\alpha\chi^\alpha\\ \bar\eta\bar\chi &:= \bar\eta_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\eta^{\dot\alpha}\bar\chi_{\dot\alpha},\end{aligned} \tag{D.9}$$

in such a way that

$$(\eta\chi)^\dagger = \bar\eta\bar\chi. \tag{D.10}$$

For anticommuting spinors one finds

$$\begin{aligned}\eta\chi &= \eta^\alpha\chi_\alpha = -\eta_\alpha\chi^\alpha = \chi^\alpha\eta_\alpha := \chi\eta\\ \bar\eta\bar\chi &= \bar\eta_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\eta^{\dot\alpha}\bar\chi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\eta^{\dot\alpha} := \bar\eta\bar\chi.\end{aligned} \tag{D.11}$$

The γ-matrices can be brought into two-component form

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\alpha} \\ (\bar\sigma^\mu)^{\dot\alpha\alpha} & \end{pmatrix}, \tag{D.12}$$

with

$$(\bar\sigma^\mu)^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}. \tag{D.13}$$

The assignment

$$\begin{aligned}\sigma^\mu &= (\sigma^0, \sigma^i) = (I_2, -\sigma^i)\\ \bar\sigma^\mu &= (\bar\sigma^0, \bar\sigma^i) = (I_2, \sigma^i)\end{aligned} \tag{D.14}$$

gives the standard representation for the γ-matrices (see appendix A). The matrix representation for the Lorentz group using γ-matrices is then

$$\begin{aligned}\Sigma^{\mu\nu} &= \frac{i}{4}[\gamma^\mu, \gamma^\nu]\\ &= \frac{i}{4}\begin{pmatrix}(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} - (\sigma^\nu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta} & 0\\ 0 & (\bar\sigma^\mu)^{\dot\alpha\alpha}(\sigma^\nu)_{\alpha\dot\beta} - (\bar\sigma^\nu)^{\dot\alpha\alpha}(\sigma^\mu)_{\alpha\dot\beta}\end{pmatrix}\\ &=: i\begin{pmatrix}(\sigma^{\mu\nu})_\alpha{}^\beta & 0\\ 0 & (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\end{pmatrix}.\end{aligned} \tag{D.15}$$

[2] $\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = -\varepsilon_{\beta\alpha} = -\varepsilon_{\dot\beta\dot\alpha}$, $\varepsilon^{12} = 1$.

So the matrices $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ are the generators of the Lorentz group in the $(\frac{1}{2},0)$ respectively the $(0,\frac{1}{2})$ representations.

The product of two spinors can be written as

$$\eta_\alpha \chi_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\eta\chi - \frac{1}{2}\varepsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma \eta^\delta (\sigma_{\mu\nu})_\delta{}^\rho \chi_\rho \tag{D.16a}$$

$$\eta_\alpha \bar{\chi}_{\dot{\alpha}} = -\frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}\eta^\delta (\sigma_\mu)_{\delta\dot{\rho}}\bar{\chi}^{\dot{\rho}}. \tag{D.16b}$$

In group theoretical language, eq. (D.16a) stands for $(0,\frac{1}{2})\otimes(0,\frac{1}{2}) = (0,0)\oplus(0,1)$, and eq. (D.16b) for $(0,\frac{1}{2})\otimes(\frac{1}{2},0) = (\frac{1}{2},\frac{1}{2})$. Using the σ-matrices one can convert a vector into a four-component spinor and vice versa

$$V_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}}V_\mu, \qquad V^\mu = -\frac{1}{2}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}V_{\alpha\dot{\alpha}}. \tag{D.17}$$

In two-component notation a Majorana spinor is written as[3]

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi} = (\psi^\alpha, \bar{\psi}_{\dot{\alpha}}). \tag{D.18}$$

For arbitrary four-component spinors one finds the relations

$$\begin{aligned} \bar{\Psi}\Phi &= \psi\phi + \bar{\psi}\bar{\phi} = \psi^\alpha\phi_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} \\ \bar{\Psi}\gamma^\mu\Phi &= \psi\sigma^\mu\bar{\phi} - \phi\sigma^\mu\bar{\psi} = \psi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} - \phi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}. \end{aligned} \tag{D.19}$$

Anticommuting Grassmann two-component spinors θ and $\bar{\theta}$ satisfy

$$\begin{aligned} \theta\theta &= \theta^\alpha\theta_\alpha = -\theta^1\theta^2 + \theta^2\theta^1 = -2\theta^1\theta^2 = -2\theta_1\theta_2 \\ \bar{\theta}\bar{\theta} &= \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = -\bar{\theta}^{\dot{2}}\bar{\theta}^{\dot{1}} + \bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}} = +2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}} = +2\bar{\theta}_{\dot{1}}\bar{\theta}_{\dot{2}}. \end{aligned} \tag{D.20}$$

This can be used to calculate the following relations

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta \\ \theta_\alpha\theta_\beta &= +\frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= +\frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \\ \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}. \end{aligned} \tag{D.21}$$

Further relations are

$$\begin{aligned} \theta^\alpha\theta^\beta\theta^\gamma &= 0 \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\gamma}} &= 0, \end{aligned} \tag{D.22}$$

and

$$\begin{aligned} (\theta\sigma^\mu\bar{\theta})\theta^\alpha &= \left(\theta^\beta(\sigma^\mu)_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right)\theta^\alpha = -\frac{1}{2}(\theta\theta)\varepsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}} = -\frac{1}{2}\theta\theta(\sigma^\mu\bar{\theta})^\alpha \\ (\theta\sigma^\mu\bar{\theta})\bar{\theta}^{\dot{\alpha}} &= \left(\theta^\beta(\sigma^\mu)_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right)\bar{\theta}^{\dot{\alpha}} = +\frac{1}{2}(\bar{\theta}\bar{\theta})\theta^\beta(\sigma^\mu)_{\beta\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}} = +\frac{1}{2}\bar{\theta}\bar{\theta}(\theta\sigma^\mu)^{\dot{\alpha}}. \end{aligned} \tag{D.23}$$

[3]Note the ambiguity in the usage of the bar-symbol.

Noting that

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\alpha} = \mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu) = 2\eta^{\mu\nu}, \tag{D.24}$$

one finds

$$(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}\eta^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}. \tag{D.25}$$

It is also true that

$$\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\lambda) = 2(\eta^{\mu\nu}\eta^{\rho\lambda} - \eta^{\mu\rho}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\rho} - i\varepsilon^{\mu\nu\rho\lambda}). \tag{D.26}$$

The Fierz rearrangement formula reads

$$\begin{aligned}(\theta\phi)(\theta\psi) &= -\frac{1}{2}\theta\theta(\phi\psi)\\ (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) &= -\frac{1}{2}\bar{\theta}\bar{\theta}(\bar{\phi}\bar{\psi}).\end{aligned} \tag{D.27}$$

The derivatives are found to be

$$\begin{aligned}\frac{\partial}{\partial\theta^\alpha}(\theta\theta) &= \left(\frac{\partial}{\partial\theta^\alpha}\theta^\beta\right)\theta_\beta - \theta_\beta\left(\frac{\partial}{\partial\theta^\alpha}\theta^\beta\right) = 2\theta_\alpha\\ \frac{\partial}{\partial\theta_\alpha}(\theta\theta) &= \left(\frac{\partial}{\partial\theta_\alpha}\theta_\beta\right)\theta^\beta - \theta^\beta\left(\frac{\partial}{\partial\theta_\alpha}\theta_\beta\right) = -2\theta^\alpha\\ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}(\bar{\theta}\bar{\theta}) &= -2\bar{\theta}_{\dot{\alpha}}\\ \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}(\bar{\theta}\bar{\theta}) &= 2\bar{\theta}^{\dot{\alpha}}.\end{aligned} \tag{D.28}$$

It is possible to introduce the concept of integration in superspace (see section 2.3). For Grassmann variables η one defines

$$\int d\eta = 0 \quad \text{and} \quad \int \eta d\eta = 1. \tag{D.29}$$

Setting the volume elements in superspace to be

$$\begin{aligned}d^2\theta &= -\frac{1}{4}d\theta^\alpha d\theta^\beta\varepsilon_{\alpha\beta}\\ d^2\bar{\theta} &= -\frac{1}{4}d\bar{\theta}_{\dot{\alpha}}d\bar{\theta}_{\dot{\beta}}\varepsilon^{\dot{\alpha}\dot{\beta}},\end{aligned} \tag{D.30}$$

one finds

$$\int (\theta\theta)d^2\theta = 1 \quad \text{and} \quad \int (\bar{\theta}\bar{\theta})d^2\bar{\theta} = 1. \tag{D.31}$$

This method allows the extraction of individual (i. e. $\theta\theta$, $\bar{\theta}\bar{\theta}$ or $\theta\theta\bar{\theta}\bar{\theta}$) components of a superfield.

Appendix E

Dimensional Analysis

The canonical dimension of

- a Dirac spinors is: $[\psi] = \frac{3}{2}$
- a scalar is: $[\phi] = 1$
- a partial derivative is: $[\partial_\mu] = 1$
- a massless vector field is: $[A_\mu] = 1$
- a mass parameter is: $[m] = 1$
- a renormalizable Lagrangian is: $[\mathcal{L}_{ren}] = 4$
- a chiral superfield is: $[\phi] = 1$
- a vector superfield is: $[V] = 0$
- an auxiliary superfield is: $[D] = [f] = 2$
- a SUSY generator is: $[Q_\alpha] = \frac{1}{2}$
- a Fayet-Iliopoulos parameter is: $[\xi^a] = 2$
- the supersymmetric parameters is: $[\epsilon] = [\theta] = [\bar{\epsilon}] = [\bar{\theta}] = -\frac{1}{2}$
- a partial derivative in anticommuting coordinates is: $[\partial_\alpha] = [\bar{\partial}_{\dot{\alpha}}] = \frac{1}{2}$

Appendix F

A Note on Supergravity and Unification

SUGRA is a step closer to the unification of the four known forces of nature — Einstein's dream of a unified field theory — than any other quantum theory of gravity. It's framework is even powerful enough to contain matter (i. e. quarks and leptons) next to all the forces. For technical reasons this feat is only self-consistently possible in 11 dimensions[1,2] (i. e. 1 time and 10 space dimensions). However, SUGRA is also plagued by some fundamental problems. It is non chiral and the largest symmetry group allowed, $O(8)$, is too small to accommodate the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ symmetry of the SM. Even worse, it yields infinite quantum corrections, i. e. a non renormalizable quantum field theory. Thus a further expansion is called for. Perturbative 10-dimensional superstring theory[3] was found to be the only framework able to avoid the hazard of infinities faced by all known quantum theories of gravity and marks theoretical physics perhaps most radical step: The abandoning of the concept of point particles in favor of 1-dimensionally extended 'strings'. Surprisingly, the theoretical approach to this string theory has unveiled an even larger overarching and unifying 11-dimensional structure, termed M-theory. Here new fundamental objects appear: p-dimensional membranes (p-branes) moving in 11-dimensional space time.

It is known that 10-dimensional SUGRA is the low-energy limit of string theory. Again, quite unexpectedly, 11-dimensional SUGRA proved to be the low-energy limit of M-theory!

[1]The appearance of higher space-time dimensions is actually not too surprising, as physicists have realized that in using higher dimensions the fundamental equations become simpler. In 1919, Theodr Kaluza showed that general relativity and electromagnetism could be unified as one 5-dimensional theory.

[2]There are less potent versions of SUGRA in 4 and 10 dimensions.

[3]The 'super' in superstrings refers to SUSY. Historically, the 1974 Wess-Zumino model of space-time SUSY (section 2.3) was presented as a generalization of the 2-dimensional worldsheet SUSY introduced in 1971 by the Ramond-Neveu-Schwarz string model. Also in 1971, Gol'fand and Likhtman independently found the super Poincaré algebra. Apparently, the first appearance of the concept of SUSY was in the context of a supergroup discovered by Myazawa in 1966. For technical details see M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory Vol. 1*, Cambridge University Press, 1987 or refs. [2] and [10].

References

[1] T. Cheng and L. Li, *Gauge Theory of Elementary Particle Physics*, Oxford Science Publications, 1996.

[2] M. Kaku, *Quantum Field Theory*, Oxford University Press, 1993.

[3] S. Dimopoulos and H. Georgi, *Nucl. Phys.* **B193** 150, 1981.

[4] L. Hall and Ch. Kolda, `hep-ph/9904236`, 1999.

[5] W. Buchmüller and D. Wyler, *Nucl. Phys.* **B268** 621, 1986.

[6] H. E. Haber in *Perspectives on Higgs Physics* edited by G. L. Kane, World Scientific, 1993.

[7] J. P. Derendinger, *Globally Supersymmetric Theories in Four and Two Dimensions*, lecture notes, ETH Zürich.

[8] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, 1992.

[9] R. N. Mohapatra, *Unification and Supersymmetry*, Springer, 1992.

[10] P. D. B. Collins, A. D. Martin and E. J. Squires, *Particle Physics and Cosmology*, Wiley, 1989.

[11] I. L. Buchbinder, S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, Institute of Physics Publishing, 1998.

[12] J. Wess and B. Zumino, *Phys. Lett.* **49 B** 52, 1974.

[13] J. F. Gunion, H. E. Haber, G. Kane, S. Dawson, *The Higgs Hunter's Guide*, Addison-Wesley, 1990.

[14] R. Barbieri in *Proceedings in Quarks, Leptons and Beyond*, Munich, 1986.

[15] J. Ellis, *Beyond the Standard Model*, `hep-ph/9812235`, 1998.